MAT251 Notes on 1.2 Factors and Multiples

**Natural numbers** $\mathbb{N}$ are the set of nonnegative integers $\{0, 1, 2, 3, ...\}$

For integers $m$ and $n$, $n$ is a **multiple of** $m$ if $n = km$, for some integer $k$.

$n$ is divisible by $m$, $m$ divides $n$, $m|n$, $m$ is a divisor of $n$, $m$ is a factor of $n$.

Else $m$ does not divide $n$, $m \nmid n$

**Example 1**

$4|12$, $5|20$, $23|23$, $11|1001$ because $1001 = 91 \cdot 11$, also $13|1001$ because $13|91$ and $91|1001$.

For nonzero $n$, $1|n$ and $n|n$, because $n = n \cdot 1 = 1 \cdot n$

When $n = km$, and $m \neq 0$, then $k$ must be the integer $\frac{n}{m}$.

Thus $m|n$ if and only if $\frac{n}{m}$ is an integer.

**Example 2**

Treat above example 1 in terms of quotients: $\frac{12}{4} = 3$

**Example 3**

Consider the special case of zero: 0 is a multiple of every $n$, but its only multiple is 0.

Divisibility $\sim$ size:

**Proposition**: If $m$ and $n$ are positive **integers** such that $m|n$, then $m \leq n$ and $\frac{n}{m} \leq n$.

**Proof**: Let $k = \frac{n}{m}$, then the integer $k$ is not negative and not zero; so $1 \leq k$ and $1 \cdot m \leq k \cdot m; m \leq n$.

Since $km = n$, we have $k|n$ and the same argument shows that $k \leq n$.

**Note**: argument fails for rationals.

**Theorem 1** An integer $n$ greater than 1 is prime iff its **only** positive divisors are 1 and $n$.

Prove the equivalent statement:

$n$ is not prime iff it has at least one positive divisor other than 1 and $n$.

If $n$ is not prime then there are integers $s$ and $t$, where $s < n$ and $t < n$ and $n = st$.

Now $s$ is a positive divisor other than 1 and $n$.

If $n$ has some positive divisor $m$ other than 1 and $n$, then $\frac{n}{m}$ is an integer $k$ and $n = k \cdot m$, and by the proposition both $k$ and $m$ are $< n$, so $n$ is not prime.

**Factoring as product of primes**: $120 = 10 \cdot 12 = 2 \cdot 5 \cdot 3 \cdot 4 = 2^3 \cdot 3 \cdot 5$. $91 = 7 \cdot 13$
1. $m|n$, “$m$ divides $n$” means there is an integer $k$ such that $n = km$.
   a) $8|4$ is false because 8 is bigger than 4     b) $4|15$ is false because 15 divided by 4 is a fraction
   c) $22|374$ is true because $374 = 17 \cdot 22$       d) $11|1001$ is true because 1001 = $91 \cdot 11$
   e) $3|1000$ is false because dividing any power of 10 by 3 leaves a remainder of 1.

2. a) $n|1$ is true only for $n = 1$   b) $n|n$ for all positive integers ($n = 1$ for all $n$)
   c) $n|n^2$ because $n^2 = n \cdot n$

3. a) The $\gcd(m, n)$ is the largest integer which is a divisor of $m$ and $n$. [Use only common factors]
   b) The $\text{lcm}(m, n)$ is the smallest integer which is a multiple of $m$ and $n$. [Use smallest common
   multiples]
   Because $27 = 3 \cdot 3 \cdot 3$ and $28 = 2 \cdot 2 \cdot 7$, $\gcd(27, 28)$ is 1 and $\text{lcm}(27, 28) = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 7 = 756$
   Because $6 = 2 \cdot 3$ and $20 = 2 \cdot 2 \cdot 5$, $\gcd(6, 20)$ is 2 and $\text{lcm}(6, 20) = 2 \cdot 2 \cdot 3 \cdot 5 = 60$
   Because $15 = 3 \cdot 5$ and $30 = 2 \cdot 3 \cdot 5$, $\gcd(15, 30)$ is $3 \cdot 5 = 15$ and $\text{lcm}(15, 30) = 2 \cdot 3 \cdot 5 = 30$
   Because $16 = 2 \cdot 2 \cdot 2 \cdot 2$ and $27 = 3 \cdot 3 \cdot 3$, $\gcd(16, 27)$ is 1 and $\text{lcm}(16, 27) = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 = 432$
   Because 13 is prime and $91 = 7 \cdot 13$, $\gcd(13, 91)$ is 13 and $\text{lcm}(13, 91) = 7 \cdot 13 = 91$.

4. Theorem 3 states that $\gcd(m, n) \cdot \text{lcm}(m, n) = m \cdot n$
   We see that $1 \cdot 756 = 27 \cdot 28$
   that $2 \cdot 60 = 6 \cdot 20$
   that $15 \cdot 30 = 15 \cdot 30$
   that $1 \cdot 432 = 16 \cdot 27$
   that $13 \cdot 91 = 13 \cdot 91$

5. a) The $\gcd(m, n)$ is the largest integer which is a divisor of $m$ and $n$. [Use only common factors]
   b) The $\text{lcm}(m, n)$ is the smallest integer which is a multiple of $m$ and $n$. [Use smallest common
   multiples]
   Because $8 = 2 \cdot 2 \cdot 2$ and $12 = 2 \cdot 2 \cdot 3$, $\gcd(8, 12)$ is 2 and $\text{lcm}(8, 12) = 2 \cdot 2 \cdot 3 = 12$
   Because $52 = 2 \cdot 2 \cdot 13$ and $96 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$, $\gcd(52, 96)$ is 2 and $\text{lcm}(52, 96) = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 = 1248$
   Because $22 = 2 \cdot 11$ and $374 = 2 \cdot 11 \cdot 17$, $\gcd(22, 374)$ is $2 \cdot 11 = 22$ and $\text{lcm}(22, 374) = 2 \cdot 11 \cdot 17 = 374$
   Because $56 = 2 \cdot 2 \cdot 7$ and $126 = 2 \cdot 3 \cdot 3 \cdot 7$, $\gcd(56, 126)$ is 2 and $\text{lcm}(56, 126) = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 7 = 504$
   We see that 37 is prime;
   $\gcd(37, 37) = 37$ and $\text{lcm}(37, 37) = 37$.

6. $33,412,363 = 4649 \cdot 7187$ so $33,412,363$ is not prime.

7. Every integer is a divisor of 0 and every multiple of 0 is 0.
   a) $\gcd(0, 10) = 10$      $\gcd(1, 10) = 1$      $\gcd(10, 10) = 10$
   b) Because $n$ is the largest divisor of itself and $n$ is a divisor of 0, $\gcd(0, n) = n$
   c) Because the only multiple of 0 is 0, $\text{lcm}(0, n) = 0$.

8. $m|n$ means $m$ divides $n$ exactly, which means there is a zero remainder, but there is not a necessary
   association with the prime number 2.