1.3 Some special sets

\( \mathbb{N} = \text{the natural numbers} = \{ 0, 1, 2, 3, 4, \ldots \} \)

\( \mathbb{P} = \text{the positive integers} = \{ 1, 2, 3, 4, \ldots \} \)

\( \mathbb{Z} = \text{the integers} = \{ 0, 1, -1, 2, -2, 3, -3, \ldots \} \)

\( \mathbb{Q} = \text{the rationals} = \{ m/n: m \in \mathbb{Z}, n \in \mathbb{Z}, n \neq 0 \} \)

\( \mathbb{R} = \text{the reals} \quad P \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \)

\( E = \text{the even naturals} = \{ n: n = 2k \text{ for some } k \in \mathbb{N} \} = \{ 0, 2, 4, 6, 8, \ldots \} = \{ 2k: k \in \mathbb{N} \} \)

\( S = \{ (-1)^n: n \in \mathbb{N} \} = \{ 1, -1 \} \)

\( I = \{ x \in \mathbb{R}: 1 \leq x < 3 \} = [1, 3), \text{ the half-closed, half-open interval between 1 and 3.} \)

\( A \subseteq B, \text{ A is a subset of B, means every element in A is also an element in B.} \)

\( A = B, \text{ A equals B, means they contain the same elements: } A \subseteq B \text{ and } B \subseteq A. \)

**Example 1:** \( \mathbb{P} \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \text{ and } S \subseteq S \text{ for any set } S. \)

\( T \subset S, \text{ T is a proper subset of S, means } T \subseteq S \text{ but } T \neq S. \)

The **empty set** has many descriptions but is denoted by \( \emptyset \) (oe) and is a subset of any set \( S \).

The **power set** of a set \( S, \mathcal{P}(S), \) is the set of all subsets of \( S. \)

**Example 2:** If a set \( S \) has \( n \) distinct elements, then \( \mathcal{P}(S) \) has \( 2^n \) elements.

Given an **alphabet**, a finite nonempty set, \( \sum \), containing unambiguous symbols called letters; a **word** is a finite string of letters from \( \sum \).

The set of all words using letters from \( \sum \) is denoted \( \sum^* \) – an infinite set and any subset of \( \sum^* \) is a **language** over \( \sum \).

**Example 3:** If the alphabet is the lower case English letters, \( \sum = \{ a, b, c, \ldots, z \} \), then \( \sum^* \) contains such words as **math, mathematics, maths, a, aa, aaaa, franc**, but **not ENGLISH**.

\( \sum^* \) is an infinite set, but the set of words in a given glossary is a finite language over \( \sum \).

If \( \sum = \{ 0, 1 \} \) then \( B = \{ 1, 10, 11, 100, 101, 110, \ldots \} \) is the set of binary representations of the positive integers.

The empty word, null word, **null string**, \( \lambda \), is a string with no letters.

**Example 4:** If \( \sum = \{ 0, 1, 2 \} \), then \( \sum^* = \{ \lambda, 0, 1, 2, 00, 01, 02, 10, 11, 12, 20, \ldots \} \)

An alphabet may contain compound letter symbols; but to avoid ambiguities in decoding words into the individual letters, an alphabet may not contain any compound letter symbols that are themselves strings that begin with one of more letters in the alphabet. For example the alphabet \( \{ a, b, Ab \} \) is acceptable, but \( \{ a, b, ac \} \) is not.

The length of a word \( w \), \( \text{length}(w) \), is a count of the letters from \( \sum \) in \( w. \)

The length (Abba) = 3 if the alphabet \( \{ a, b, Ab \} \) is used.
1. List five elements in each of the following sets:
   a) \( \{ n \in \mathbb{N} : n \text{ is divisible by 5} \} = \{0, 5, 10, 15, 20, \ldots \} \)
   b) \( \{2n+1 : n \in \mathbb{P}\} = \{3, 5, 7, 9, 11, \ldots \} \)
   c) \( \varnothing \{1, 2, 3, 4, 5\}\) has 32 elements, which are subsets of \( \{1, 2, 3, 4, 5\} \). There is 1 empty set, 5 sets with one element, 10 subsets with 2 elements, 10 subsets with 3 elements, 5 subsets with 4 elements, and 1 (sub)set with 5 elements. Some examples: \( \varnothing, \{2\}, \{3,5\}, \{1, 2, 4\}, \{1, 2, 3, 5\}, \{1, 2, 3, 4, 5\} \).
   d) \( \{2^n : n \in \mathbb{N}\} = \{1, 2, 4, 8, 16, \ldots \} \)
   e) \( \{1/n : n \in \mathbb{P}\} = \{1, 1/2, 1/3, 1/4, 1/5, \ldots \} \)
   f) \( \{r : r \in \mathbb{Q}, 0 < r < 1\} = \{1/2, 1/3, 1/4, 2/3, 1/5, \ldots \} \)
   g) \( \{n \in \mathbb{N} : n + 1 \text{ is prime}\} = \{1, 2, 4, 6, \ldots \} \)

2. List the elements in the following sets.
   a) \( \{1/n : n = 1, 2, 3, 4\} = \{1, 1/2, 1/3, 1/4\} \)
   b) \( \{n^2 - n : n = 0, 1, 2, 3, 4\} = \{0, 0, 2, 6, 12\} = \{0, 2, 6, 12\} \)
   c) \( \{1/n^2 : n \in \mathbb{P}, n \text{ is even and } n < 11\} = \{1/4, 1/16, 1/36, 1/64, 1/100\} \)
   d) \( \{2 + (-1)^n : n \in \mathbb{N}\} = \{3, 1, 3, 1, \ldots\} = \{1, 3\} \)

3. List (at least) five elements in each of the following sets.
   a) If \( \{a, b, c\} \), then \( \{a, b, c, aa, ab, ac, ba, bb, bc, ca, cb, cc, aaa, \ldots\} \)
   b) If \( \{a, b\} \), then \( \{w \in \sum^* : \text{length}(w) = 2\} = \{a, a, b, a, ab, ba, bb\} \)
   c) If \( \{a, b\} \), then \( \{w \in \sum^* : \text{length}(w) = 4\} = \{aaaa, aaab, aaba, aabb, \ldots bbbb\} \)
      Only a) and b) contain the empty word \( \lambda \).

4. List the following sets if they are nonempty; write \( \varnothing \) for an empty set.
   a) \( \{n \in \mathbb{N} : n^2 = 9\} = \{3\} \)
   b) \( \{n \in \mathbb{Z} : n^2 = 9\} = \{-3, 3\} \)
   c) \( \{x \in \mathbb{R} : x^2 = 9\} = \{-3, 3\} \)
   d) \( \{n \in \mathbb{N} : 3 < n < 7\} = \{4, 5, 6\} \)
   e) \( \{n \in \mathbb{Z} : 3 < |n| < 7\} = \{-6, -5, -4, 4, 5, 6\} \)
   f) \( \{x \in \mathbb{R} : x^2 < 0\} = \varnothing \)

5. List the following sets if they are nonempty; write \( \varnothing \) for an empty set.
   a) \( \{n \in \mathbb{N} : n^2 = 3\} = \varnothing \)
   b) \( \{x \in \mathbb{Q} : x^2 = 3\} = \varnothing \)
   c) \( \{x \in \mathbb{R} : x < 1 \text{ and } x \geq 2\} = \varnothing \)
   d) \( \{3n + 1 : n \in \mathbb{N} \text{ and } n \leq 6\} = \{1, 4, 7, 10, 13, 16, 19\} \)
   e) \( \{n \in \mathbb{P} : n \text{ is prime and } n \leq 15\} = \{2, 3, 5, 7, 11, 13\} \)

6. List the following sets if they are nonempty; write \( \varnothing \) for an empty set.
   a) \( \{n \in \mathbb{N} : n\|12\} = \{1, 2, 3, 4, 6, 12\} \)
   b) \( \{n \in \mathbb{N} : n^2 + 1 = 0\} = \varnothing \)
   c) \( \{n \in \mathbb{N} : \left\lfloor \frac{n}{3} \right\rfloor = 8\} = \{24, 25, 26\} \)
   d) \( \{n \in \mathbb{N} : \left\lfloor \frac{n}{2} \right\rfloor = 8\} = \{15, 16\} \)
7. Let $A = \{ n \in \mathbb{N} : n \leq 20 \} = \{ 0, 1, 2, 3, ..., 19, 20 \}$ List the following sets if they are nonempty; write $\emptyset$ for an empty set.
   a) $\{ n \in A : 4 \mid n \} = \{ 0, 4, 8, 12, 16, 20 \}$
   b) $\{ n \in A : n \mid 4 \} = \{ 1, 2, 4 \}$
   c) $\{ n \in A : \max \{ n, 4 \} = 4 \} = \{ 0, 1, 2, 3, 4 \}$
   d) $\{ n \in A : \max \{ n, 14 \} = n \} = \{ 14, 15, 16, 17, 18, 19, 20 \}$

8. How many elements are there in the following sets? Write $\infty$ if the set is infinite.
   a) $|\{ n \in \mathbb{N} : n^2 = 2 \}| = |\emptyset| = 0$
   b) $|\{ n \in \mathbb{Z} : 0 \leq n \leq 73 \}| = |\{ 0, 1, 2, ..., 73 \}| = 74$
   c) $|\{ n \in \mathbb{Z} : 5 \leq |n| \leq 73 \}| = |\{-73, -72, ..., -6, 5, 6, ..., 72, 73 \}| = 2(73 - 5 + 1) = 138$
   d) $|\{ n \in \mathbb{Z} : 5 < n < 73 \}| = |\{ 6, 7, ..., 72 \}| = 72 - 6 + 1 = 67$
   e) $|\{ n \in \mathbb{Z} : n \text{ is even and } |n| \leq 73 \}| = \{ -72, -70, ..., 0, ..., 72 \}| = 36 + 1 + 36 = 73$
   f) $|\{ x \in \mathbb{Q} : 0 \leq x \leq 73 \}| = \infty$
   g) $|\{ x \in \mathbb{Q} : x^2 = 2 \}| = |\emptyset| = 0$
   h) $|\{ x \in \mathbb{R} : x^2 = 2 \}| = |\{-\sqrt{2}, \sqrt{2}\}| = 2$

9. How many elements are there in the following sets? Write $\infty$ if the set is infinite.
   a) $|\{ x \in \mathbb{R} : 0.99 < x < 1.00 \}| = \infty$
   b) $|\wp(\{ 0, 1, 2, 3 \})| = 2^4 = 16$
   c) $|\wp(\mathbb{N})| = \infty$
   d) $|\{ n \in \mathbb{N} : n \text{ is even} \}| = \infty$
   e) $|\{ n \in \mathbb{N} : n \text{ is prime} \}| = \infty$
   f) $|\{ n \in \mathbb{N} : n \text{ is even and prime} \}| = |\{ 2 \}| = 1$
   g) $|\{ n \in \mathbb{N} : n \text{ is even or prime} \}| = |\{ 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 16, ... \}| = \infty$

11. Which of the following sets are subsets of which? [Consider all 16 possibilities.]
   A = $\{ n \in \mathbb{P} : n \text{ is odd} \} = \{ 1, 3, 5, 7, 9, ... \}$
   B = $\{ n \in \mathbb{P} : n \text{ is prime} \} = \{ 2, 3, 5, 7, 11, ... \}$
   C = $\{ 4n+3 : n \in \mathbb{P} \} = \{ 7, 11, 15, ... \}$
   D = $\{ x \in \mathbb{R} : x^2 - 8x + 15 = 0 \} = \{ x \in \mathbb{R} : (x-3)(x-5) = 0 \} = \{ 3, 5 \}$
   $A \subseteq B$; but it is not a subset of the others.
   $B \subseteq C$; but it is not a subset of the others.
   $C \subseteq A$ and $C \subseteq C$; but not a subset of the others.
   $D \subseteq A$ and $D \subseteq B$ and $D \subseteq D$, but $D \not\subseteq C$.
   [Note the faulty punctuation in the text may be misleading.]

13. Consider the three alphabets: $\Sigma_1 = \{ a, b, c \}$
    $\Sigma_2 = \{ a, b, ca \}$
    $\Sigma_3 = \{ a, b, Ab \}$
    Determine to which of $\Sigma_1^*, \Sigma_2^*, \Sigma_3^*$ each word below belongs and give its length in each language.
    a) $aba \in \Sigma_1^*$ and has length 3.
    b) $bAb \in \Sigma_2^*$ and has length 2.
    c) $cba \in \Sigma_1^*$ and has length 3.
    d) $cab \in \Sigma_1^*$ of length 3; and $\in \Sigma_2^*$ and has length 2.
    e) $caab \in \Sigma_1^*$ of length 4; and $\in \Sigma_2^*$ and has length 3.
    f) $baAb \in \Sigma_3^*$ and has length 3.