A **function** $f$ assigns to each element $x$ in some set $S$ a unique element in a set $T$. $f$ is defined on $S$ with values in $T$. The set $S$ is called the **domain** of $f$, $\text{Dom}(f)$; $f(x)$ is called the **image** of $x$ under $f$; the set of all images is the **image** of $f$, $\text{Im}(f)$. Note: although it is interesting to know the image sets, they are not needed in the definition. A set that does contain the image set is called a **codomain**. $f : S \rightarrow T$ maps $S$ into $T$.

**Examples:**

$g(n) = n^3 - 73n + 5$  $g : \mathbb{N} \rightarrow \mathbb{Z}$

absolute value: $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$  $F(x) = |x|$, maps $\mathbb{R}$ to $[0, \infty)$

floor function: $[x]$ maps $\mathbb{R}$ to $\mathbb{Z}$.

$$f(m, n) = \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{m-1}{2} \right\rfloor$$

The **graph** of $f : S \rightarrow T$ is $\{(x, y) \in S \times T : y = f(x)\}$

Then we get the **formal definition**: A function with domain $S$ and codomain $T$ is a subset $G$ of $S \times T$ for which for each $x \in S$ there is exactly one $y \in T$ such that $(x, y) \in G$.

**Functions with image \{0, 1\}**

**Characteristic function**: If $A$ is a subset of $S$, then the characteristic function of $A$

$$\chi_A (x) = \begin{cases} 1 & \text{for } x \in A \\ 0 & \text{for } x \in S \setminus A \end{cases}$$  See pages 39 & 40 for discussion of **one-to-one** and **onto**.

$$f_2(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$ is the characteristic function $\chi_{[0, \infty)}$ defined on $\mathbb{R}$.

$G(n) =$ remainder when $n$ is divided by 2, is characteristic function for odd natural numbers.

**Function composition**: $f : S \rightarrow T$ and $g : T \rightarrow U$ $(g \circ f)(x) = g(f(x))$ for all $x$ in $S$.

Example: $f(x) = x^2$ and $g(x) = x+1$.

$g(f(x)) = g(x^2) = x^2 + 1$ and $f(g(x)) = f(x+1) = (x+1)^2 = x^2 + 2x + 1$

Composition is not commutative, but can be shown to be associative.
Exercises 1.5 page 33

1. \( f(n) = n^2 + 3, \ g(n) = 5n - 11 \) for \( n \in \mathbb{N} \). \( f : \mathbb{N} \to \mathbb{N}, \ g : \mathbb{N} \to \mathbb{Z} \)
   a) \( f(1) = 4, \ g(1) = -6 \)
   b) \( f(2) = 7, \ g(2) = -1 \)
   c) \( f(3) = 12, \ g(3) = 4 \)
   d) \( f(4) = 19, \ g(4) = 9 \)
   e) \( f(5) = 28, \ g(5) = 14 \)
   f) \( f(n) + g(n) = n^2 + 3 + 5n - 11 = n(n+5) - 8 \). Now for any \( n \), either \( n \) is even or \( (n+5) \) is even, so \( f(n) + g(n) \) is even.

2. \( h : \mathbb{P} \to \mathbb{P}, \ h(n) = |\{k \in \mathbb{N} : k|n\}| \) for \( n \in \mathbb{P} \).
   \( h(1) = 1 \)
   \( h(2) = |\{1,2\}| = 2 \)
   \( h(3) = |\{1,3\}| = 2 \)
   \( h(4) = |\{1,2,4\}| = 3 \)
   \( h(5) = |\{1,5\}| = 2 \)
   \( h(6) = |\{1,2,3,6\}| = 4 \)
   \( h(7) = |\{1,7\}| = 2 \)
   \( h(8) = |\{1,2,4,8\}| = 4 \)
   \( h(9) = |\{1,3,9\}| = 3 \)
   \( h(10) = |\{1,2,5,10\}| = 4 \)
   \( h(73) = |\{1,73\}| = 2 \)

3. \( \Sigma^* = \) the set of all words formed from the alphabet \( \Sigma = \{a, b\} \)
   a) \( \text{length}(bab) = 3 \)
   b) \( \text{length}(aaaaaaa) = 8 \)
   c) \( \text{length}(\lambda) = 0 \)
   d) \( \text{Im}(\text{length}) = \mathbb{N}; \) consider a string of \( n \) a’s.

4. \( \Sigma^* = \) the set of all words formed from the alphabet \( \Sigma = \{a, b\} \)
   \( g(n) = \{w \in \Sigma^* : \text{length}(w) \leq n\} \) for \( n \in \mathbb{N} \). \( g : \mathbb{N} \to \mathcal{P}(\Sigma^*) \).
   a) \( g(0) = \{\lambda\} \), the set containing only the empty word.
   b) \( g(1) = \{\lambda, a, b\} \)
   c) \( g(2) = \{\lambda, a, b, aa, ab, ba, bb\} \)
   d) \( |g(n)| = \sum_{k=0}^{n} 2^k = 2^{n+1} - 1 \), a finite number.
   e) Every set \( g(n) \) is a subset of \( \mathcal{P}(\Sigma^*) \), but the set \( \{a\} \) is a set in \( \Sigma^* \) that is not in \( \text{Im}(g) \).
      Also \( \{\lambda, a, aa, aaaa, \ldots\} \) is not in \( \text{Im}(g) \).

5. \( f(m, n) = \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{m - 1}{2} \right\rfloor \). \( F : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \)
   a) \( f(0, 0) = 0 - [ -0.5 ] = 0 - (-1) = 1 \)
   \( f(8, 8) = [4] - [3.5] = 4 - (3) = 1 \)
   \( f(8, -8) = [-4] - [-4.5] = (-4) - (-5) = 1 \)
   \( f(73, 73) = [36.5] - [36] = 36 - 36 = 0 \)
   b) If \( n \) is even, say \( n = 2k \), then \( f(2k, 2k) = |k| - |k - \frac{1}{2}| = k - (k - 1) = 1 \)
      If \( n \) is odd, say \( n = 2k+1 \), then \( f(2k+1, 2k+1) = |k + \frac{1}{2}| - |k| = k - k - 0 \)
      Note that the function \( f(n, n) \) is a \textit{characteristic function} for the set of even integers.
6. On $\mathbb{P} \times \mathbb{P}$, $\gcd(m, n)$ is the greatest common divisor of $m$ and $n$.
   a) $\gcd(7, 14) = 7; \quad \gcd(14, 28) = 14; \quad \gcd(1001, 2002) = 1001$
   b) $\gcd(n, 2n) = n$ for all $n \in \mathbb{P}$
   c) $\Im(\gcd) = \mathbb{P}$ because $\gcd(n, n) = n$.

7. $f: \mathbb{R} \times \mathbb{R}$ defined by
   a) $f(3) = 3^3 = 27$
   $f(\sqrt[3]{2}) = \sqrt[3]{8} = 2$
   $f(-\sqrt[3]{2}) = -(-\sqrt[3]{8}) = 2$
   b) See graph at right.
   c) $\Im(f) = \{x \in \mathbb{R} : x \geq 0\} = [0, \infty)$

8. $S = \{1, 2, 3, 4, 5\}; \quad l_S(n) = n, f(n) = 6 - n, g(n) = \max\{3, n\}, h(n) = \max\{1, n-1\}$
   a) $l_S = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$
   $f = \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\}$
   $g = \{(1, 3), (2, 3), (3, 3), (4, 4), (5, 5)\}$
   $h = \{(1, 1), (2, 1), (3, 2), (4, 3), (5, 4)\}$
   b) $5 \circ \circ \circ \circ \bullet \quad 5 \circ \circ \circ \circ \circ \quad 5 \circ \circ \circ \circ \bullet \quad 5 \circ \circ \circ \circ \circ$
   $4 \circ \circ \circ \bullet \quad 4 \circ \circ \circ \circ \circ \quad 4 \circ \circ \circ \circ \bullet \quad 4 \circ \circ \circ \circ \circ$
   $3 \circ \circ \circ \circ \quad 3 \circ \circ \circ \circ \circ \quad 3 \circ \circ \circ \bullet \quad 3 \circ \circ \circ \bullet \quad 3 \circ \circ \circ \circ \circ$
   $2 \circ \circ \circ \circ \quad 2 \circ \circ \circ \circ \circ \quad 2 \circ \circ \circ \bullet \quad 2 \circ \circ \circ \bullet \quad 2 \circ \circ \circ \circ \circ$
   $1 \circ \circ \circ \circ \quad 1 \circ \circ \circ \circ \circ \quad 1 \circ \circ \circ \bullet \quad 1 \circ \circ \circ \bullet \quad 1 \circ \circ \circ \circ \circ$
   $1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5$
   $l_S \quad f \quad g \quad h$

9. $n \in \mathbb{Z}, f(n) = \frac{1}{2}(-1)^n + 1$
   $f(0) = \frac{1}{2}(-1)^0 + 1 = \frac{1}{2}[2] = 1 \quad f(1) = \frac{1}{2}(-1)^1 + 1 = \frac{1}{2}[0] = 0$
   $f(2) = \frac{1}{2}(-1)^2 + 1 = \frac{1}{2}[2] = 1 \quad f(-1) = \frac{1}{2}(-1)^{-1} + 1 = \frac{1}{2}[0] = 0$
   $f$ is a characteristic function for $\{n \in \mathbb{Z} : n \text{ is even}\}$

10. For any set $S$ with subsets $A$ and $B$,
    a) $\chi_A \cdot \chi_B$ is 1 if and only if each factor is 1, so it is characteristic function for $A \cap B$.
    b) $\chi_A + \chi_B - \chi_{A \cap B}$ is 1 for elements only in $A$, or only in $B$, or in both, so ... $A \cup B$.
    c) $\chi_A + \chi_B - 2\chi_{A \cap B}$ is 1 for elements only in $A$, or only in $B$, so ... $A \oplus B$.

11. a) $f(n) = \left[\frac{n}{2}\right] + \left[\frac{n}{3}\right]$ for $n \in \mathbb{N}$.
    $f(0) = 0+0 = 0; \quad f(1) = 0+0 = 0; \quad f(2) = 1+0 = 1; \quad f(3) = 1+1 = 2;$
    $f(4) = 2+1 = 3; \quad f(5) = 2+1 = 3; \quad f(6) = 3+2 = 5; \quad f(7) = 3+2 = 5$
    $f(8) = 4+2 = 6; \quad f(9) = 4+3 = 7; \quad f(10) = 5+3 = 8; \quad f(73) = 36+24 = 60.$
    b) $g(n) = \left[\frac{n}{2}\right] - \left[\frac{n}{2}\right]$ for $n \in \mathbb{Z}$.
    $g(0) = 0 - 0 = 0; \quad g(1) = 1 - 0 = 1; \quad g(-1) = 0 - (-1) = 1; \quad g(2) = 1 - 1 = 0,$ ...
    $g$ is characteristic function for $\{n \in \mathbb{Z} : n \text{ is odd}\}$. 