Recursive definitions

A sequence is defined recursively if (B) some finite set of values is specified and (R) the remaining values are defined in terms of the previous values by a recurrence formula. A recurrence relation is a formula that defines each number in a sequence of numbers in terms of previous numbers in the sequence. Such relations often describe (dynamical) systems that evolve over time, as well as determining the number of steps in algorithmic computations.

An example of a recurrence relation is \( a_n = 3a_{n-1} \), for \( n \geq 1 \).
If the starting point \( a_0 \) is given as 2, then the sequence is 2, 6, 18, 54, 162, . . .
Or a starting point of \( a_0 = 5 \) would produce 5, 15, 45, 135, 405, . . .

The Fibonacci sequence given by \( F_n = F_{n-1} + F_{n-2} \), for \( n \geq 2 \), with \( F_0 = 0, F_1 = 1 \), produces the sequence: 0, 1, 1, 2, 3, 5, 8, 13, ...

Note: some applications do not include \( F_0 \)

Values in a recursively-defined sequence can be determined either iteratively – finding all values prior to the desired term or recursively – by calculating only those preceding terms on which the desired term depends.

A later lesson will illustrate how simple recurrence relations can be solved to produce explicit formulas used to calculate specific values in the sequence of values.

The examples cited above are iterative calculations to determine all the elements in the sequence up to a certain point. An alternative method to calculate a particular element, say the seventh element, would be to use the recurrence relation recursively to calculate (only the necessary) preceding values until the stopping condition [subscript 0] is reached. Either approach easily yields an explicit formula \( a_n = a_03^n \) for \( n \geq 1 \). This is a simple geometric sequence.

A more interesting example is \( b_n = 3b_{n-2} \), for \( n \geq 2 \), with \( b_0 = 2 \) and \( b_1 = 5 \). The resulting sequence can be calculated iteratively as 2, 5, 6, 15, 18, 45, . . . A recursive calculation would depend on whether the subscript was even or odd (to determine the appropriate stopping point). Also an explicit formula would have two parts: \( b_{2k} = 2 \cdot 3^k \) and \( b_{2k+1} = 5 \cdot 3^k \) for \( k \) in the Naturals.

OTHER EXAMPLES

\[ \text{SUM}(0)=0; \text{SUM}(n+1)=(n+1)\text{+SUM}(n) \quad \{\text{Gives subtotals up to n}\} \]

\[ \text{FACT}(0)=1; \text{FACT}(n+1)=(n+1)\text{FACT}(n) \quad \text{for natural n} \quad \{\text{Equivalent to n!}\} \]

\[ \text{SEQ}(0)=1; \text{SEQ}(n+1)=(n+1)/\text{SEQ}(n) \quad \text{for natural n.} \quad \{\text{There is only one such sequence.}\} \]

AN UNUSUAL EXAMPLE

\( L(1)=1; \ L(n) = 2 \cdot L(\lfloor n/2 \rfloor) \) for \( n \geq 2 \), where \( \lfloor x \rfloor \) denotes the greatest integer not greater than \( x \).
Note that calculating \( L(17) \) recursively only requires four steps if done recursively.
1. We define \( s_0 = 1 \) and \( s_{n+1} = 2/s_n \) for \( n \in \mathbb{N} \).
   a) List the first five terms of this sequence: 1, 2, 1, 2, 1, ...
   b) What is the set of values of \( s_n \)? \( \{s_n\} = \{1, 2\} \).

2. We recursively define \( \text{SEQ}(0) = 0 \) and \( \text{SEQ}(n+1) = 1/[1+\text{SEQ}(n)] \) for \( n \geq 0 \).
   Calculate \( \text{SEQ}(n) \) for \( n = 1, 2, 3, 4, 6 \).
   \( \text{SEQ}(1) = 1/[1+0] = 1 \);
   \( \text{SEQ}(2) = 1/[1+1/2] = 2/3 \);
   \( \text{SEQ}(4) = 1/[1+3/5] = 3/7 \);
   \( \text{SEQ}(6) = 1/[1+3/8] = 8/13 \).

3. Consider the sequence \((1, 3, 9, 27, 81, ...)\).
   a) Give a formula for the \( n \)th term \( \text{SEQ}(n) \) where \( \text{SEQ}(0) = 1 \).
      \( \text{SEQ}(n) = 3^n \) for \( n \geq 0 \).
   b) Give a recursive for definition \( \text{SEQ}(n) \):
      \( \text{SEQ}(0) = 1; \text{SEQ}(n+1) = 3 \text{SEQ}(n) \) for \( n \geq 0 \).

4a) Five a recursive definition for the sequence \((2, 2^2, (2^2)^2, ((2^2)^2)^2, ...)\).
   \( \text{SQR}(0) = 2; \text{SQR}(n+1) = \text{SQR}(n)^2 \) for \( n \geq 0 \).

b) Give a recursive definition for the sequence \((2, 4, 16, 65536, ...)\).
   \( \text{POW}(0) = 2; \text{POW}(n+1) = 2^{\text{POW}(n)} \) for \( n \geq 0 \).

5. Is the following a recursive definition for a sequence \( \text{SEQ} \)? Explain.
   \( \text{SEQ}(0) = 1; \text{SEQ}(n+1) = \text{SEQ}(n)/(100-n) \).
   No, (R) only works for \( 0 \leq n < 100 \); but it does determine a finite sequence.

6a) Calculate \( \text{SEQ}(9) \) where \( \text{SEQ}(0) = 1; \text{SEQ}(n+1) = (n+1)/\text{SEQ}(n) \) for \( n \in \mathbb{N} \).
   \( \text{SEQ}(9) = 9/(8/(7/(6/(5/(4/(3/(2/1))))))) \) equals \( 315/128 \).

b) Calculate \( \text{FIB}(12) \) where \( \text{FIB}(1) = \text{FIB}(2) = 1; \text{FIB}(n) = \text{FIB}(n-1) + \text{FIB}(n-2) \) for \( n \geq 3 \).
   \( \text{FIB}(12) = \text{FIB}(11) + \text{FIB}(10) = [\text{FIB}(10) + \text{FIB}(9)] + \text{FIB}(10) \); that is, \( 2\text{FIB}(10) + \text{FIB}(9) \),
   \( \text{FIB}(10) \) which equals \( 5\text{FIB}(9) + 2\text{FIB}(8) \) which equals \( 5\text{FIB}(7) + 3\text{FIB}(6) \) which equals \( 8\text{FIB}(7) + 5\text{FIB}(6) \) which equals \( 13\text{FIB}(6) + 8\text{FIB}(5) \) which equals \( 21\text{FIB}(5) + 13\text{FIB}(4) \) which equals \( 34\text{FIB}(4) + 21\text{FIB}(3) \) which equals \( 55\text{FIB}(3) + 34\text{FIB}(2) \) which equals \( 89\text{FIB}(2) + 55\text{FIB}(1) = 89 + 55 = 144 \).

c) Calculate \( \text{Q}(19) \) where \( \text{Q}(1) = 1; \text{Q}(n) = 2\text{Q}([n/2]) + n \) for \( n \geq 2 \).
   \( \text{Q}(19) = 2\text{Q}(9) + 19 = 2[2\text{Q}(4) + 9] + 19 \); that is, \( 2^2\text{Q}(4) + 2\cdot9 + 19 \)
   which equals \( 2^2[2\cdot\text{Q}(2) + 4] + 2\cdot9 + 19 \); that is \( 2^3\text{Q}(2) + 2^2\cdot4 + 2\cdot9 + 19 \)
   which equals \( 2^3[2\cdot\text{Q}(1) + 2] + 2^2\cdot4 + 2\cdot9 + 19 \); that is, \( 2^4(1) + 2^32 + 2^2\cdot4 + 2\cdot9 + 19 = 85 \).

7. Let \( \sum \{a, b, c \} \) and let \( s_n \) denote the number of words of length \( n \) that do not contain the string \( aa \).
   a) Calculate \( s_0, s_1, \) and \( s_2 \).
   \( \{\lambda, a, b, c, ab, ac, ba, bb, bc, ca, cb, cc, \ldots\} \).
   \( s_0 = 1; s_1 = 3; \) and \( s_2 = 8 \).
   b) Find a recurrence formula for \( s_n \).
   We can adjoin either \( b \) or \( c \) onto each of the \( s_{n-1} \) words but we can adjoin \( a \) only onto those \( 2s_{n-2} \)
   words that did not end in a [that is, previously had adjoined \( b \) or \( c \)].
   Hence \( s_n = 2(s_{n-1}) + 1(2s_{n-2}) \) for \( n \geq 2 \).
   c) Calculate \( s_3 \) and \( s_4 \).
   \( s_3 = 2(8) + 2(3) = 22 \) and \( s_4 = 2(22) + 2(8) = 60 \).
8. Let $\Sigma = \{a, b\}$ and let $s_n$ denote the number of words of length $n$ that do not contain the string $ab$.
   a) Calculate $s_0, s_1, s_2,$ and $s_3$. \{\lambda, a, b, aa, ba, bb, aab, aba, bba, bbb, \ldots \} \ s_0 = 1; \ s_1 = 2; \ s_2 = 3; \ s_3 = 4.$
   b) Find a formula for $s_n$ and prove it is correct. $s_n = n+1$. Proof is by induction on $n$.
   Basis is clearly true as $0 + 1 = 1 = s_0$.
   The induction step follows because if $s_k = k+1$ we can adjoin $a$ to each of these $(k + 1)$ words, and we can adjoin $b$ only to the one word of length $k$ that did not end in $a$; so $s_{k+1} = (k+1) + 1$.

9. Let $\Sigma = \{a, b\}$ and let $t_n$ denote the number of words of length $n$ with an even number of $a$’s.
   a) Calculate $t_0, t_1, t_2,$ and $t_3$. \{\lambda, b, aa, bb, aab, aba, bba, bbb, \ldots \} \ t_0 = 1; \ t_1 = 1; \ t_2 = 2; \ t_3 = 4.$
   b) Find a formula for $t_n$ and prove it is correct. $t_n = 2^{n-1}$ for $n \geq 1$

   For $r \geq 1$, we can adjoin $b$ to words in $\Sigma^{r-1}$ with an even number of $a$’s and adjoining $a$ to words in $\Sigma^{r-1}$ with an odd number of $a$’s. Thus, $t_r = t_{r-1} + (2^{r-1} - t_{r-1})$, which equals $2^{r-1}$.
   No induction needed.
   c) We remark that the formula fails for $r = 0$; because $2^{-1} \neq 1$.

10. Consider the sequence defined by
   (B) $SEQ(0) = 1$, $SEQ(1) = 0$
   (R) $SEQ(n) = SEQ(n - 2)$ for $n \geq 2$.
   a) List the first few terms of this sequence. \(1, 0, 1, 0, 1, \ldots\)
   b) What is the set of values of this sequence? \{0, 1\}