Proof by mathematical induction using a strong hypothesis

Occasionally a proof by mathematical induction is made easier by using a **strong hypothesis**:

To show \( P(n) \) [a **statement form** that depends on variable \( n \)], do:

1. **BASIS**: Establish \([P(m) \land P(m+1) \land \ldots \land P(m+r)]\) is true for some initial value of \( m \).
2. **IMPLICATION**: Prove \([P(m) \land P(m+1) \land \ldots \land P(m+r) \land \ldots \land P(k)] \Rightarrow P(k+1)\) for any \( k \geq m+r \).

Then by **strong** induction, we know \( P(n) \) is true for every \( n \geq m \).

Note that \( r = 0 \) gives the simple principle of induction. Also it can be shown that the principle of strong induction follows from simple induction.

**Example**:

Consider a sequence defined by \( a_0 = 1, a_1 = 2, a_2 = 3; \) and \( a_n = a_{(n-2)} + 2a_{(n-3)} \) for \( n \geq 3 \).

We compute the next five values as follows:

\[
a_3 = 2 + 2(1) = 4, \quad a_4 = 3 + 2(2) = 7, \quad a_5 = 4 + 2(3) = 10, \quad a_6 = 7 + 2(4) = 15, \quad a_7 = 10 + 2(7) = 24.
\]

It appears that, after the first term, the values are increasing at a rate of about 150\%. We may establish by induction that \( P(n) : a_n > (3/2)^n \) for \( n \geq 1 \).

Because the definition of the general term requires knowledge not just of the term immediately preceding, but of terms two and three preceding, it is useful to use a **strong hypothesis** for induction.

**BASIS**: \( a_1 = 2 > 1.5 = (3/2) \) so \( P(1) \) is true.

\[
a_2 = 3 > 2.25 = (3/2)^2 \) so \( P(2) \) is true.
\]

\[
a_3 = 4 > 3.375 = (3/2)^3 \) so \( P(3) \) is true.
\]

**INDUCTION**: Given \( P(j) \) is true for \( 1 \leq j \leq k \) where \( k \geq 3 \); that is, \( a_j > (3/2)^j \), we establish the desired result by a sequence of substitutions and inequalities.

\[
\text{then } a_{(k+1)} = a_{(k-1)} + 2(a_{(k-2)}) \text{ and by the strong hypothesis this must be,}
\]

\[
> (3/2)^{(k-1)} + 2((3/2)^{(k-2)}) \text{ and factoring out } (3/2)^{(k-2)} \text{ this}
\]

\[
= (3/2)^{(k-2)][(3/2) + 2] \text{ which}
\]

\[
= (3/2)^{(k-2)][3.5] \text{ which}
\]

\[
> (3/2)^{(k-2)][3.375] \text{ which}
\]

\[
= (3/2)^{(k-2)][27/8] \text{ which}
\]

\[
= (3/2)^{(k+1)} \) and the result is established.
\]

Hence by strong induction \( P(n) \) is true for all \( n \geq 1 \).

**Another example**:

Consider the sequence defined by \( a_0 = 1, a_1 = 2; \) and \( a_n = \frac{(a_{(n-1)})^2}{a_{(k-2)}} \) for \( n \geq 2 \).

We compute the next three values as follows:

\[
a_2 = 2^2/1 = 4, \quad a_3 = 4^2/2 = 8, \quad a_4 = 8^2/4 = 16.
\]

It appears that each term is a power of 2. We may establish by induction that \( P(n) : a_n = 2^n \) for \( n \geq 0 \).

**BASIS**: \( a_0 = 1 = 2^0 \), and \( a_1 = 2 = 2^1 \) so \( P(0) \) and \( P(1) \) are both true.

**INDUCTION**: Given \( P(j) \) is true for \( 0 \leq j \leq k \) where \( k \geq 1 \); that is, \( a_j = 2^j \), then

\[
a_{(k+1)} = \left(\frac{a_{(k+1-1)}}{a_{(k+1-2)}}\right)^2 = \frac{(2^k)^2}{2^{(k-1)}} = 2^{(k+1)} \) and the result is established.
\]

Hence by strong induction \( P(n) \) is true for all \( n \geq 0 \).
1. Prove $3 + 11 + \ldots + (8n - 5) = 4n^2 - n$ for $n \in \mathbb{P}$; that is, $p(n) = \sum_{i=1}^{n} (3i - 5) = 4n^2 - n$ for $n \in \mathbb{P}$.

Only First Principle is needed: (B): $(8\cdot1 - 5) = 3 = 4 - 1 = 4\cdot1^2 - 1$ so $p(1)$ is true.

(I): Given $p(k - 1)$; that is, $\sum_{i=1}^{k-1} (3i - 5) = 4(k - 1)^2 - (k - 1)$ then $\sum_{i=1}^{k} (3i - 5) = \left[4(k - 1)^2 - (k - 1)\right] + (3k - 5)$ which equals $4k^2 - 8k + 4 - k + 1 + 8k - 5 = 4k^2 - k$, so $p(k - 1)$ implies $p(k)$.

By (B) & (I), $p(n)$ is true for all $n \in \mathbb{P}$.

2. For $n \in \mathbb{P}$, prove:

a) $1 \cdot 2 + 2 \cdot 3 + \cdots + n(n + 1) = \frac{1}{3}n(n + 1)(n + 2)$; that is, $p(n) = \sum_{i=1}^{n} i(i + 1) = \frac{1}{3}n(n + 1)(n + 2)$

Only First Principle is needed: (B): $1(1+1) = 2 = \frac{1}{3}1 \cdot 2 + 2 \cdot 3 = \frac{1}{3}1(1+1)(1+2)$ so $p(1)$ is true.

(I): Given $p(k - 1)$; that is, $\sum_{i=1}^{k-1} i(i + 1) = \frac{1}{3}(k - 1)((k - 1) + 1)((k - 1) + 2) = \frac{1}{3}(k - 1)(k)(k + 1)$, then

$$
\sum_{i=1}^{k} i(i + 1) = \left[\frac{1}{3}(k - 1)(k)(k + 1)\right] + k(k + 1) \text{ which equals } \frac{1}{3}(k+1)(k+2) \text{ so } p(k - 1) \text{ implies } p(k). \text{ By (B) & (I), } p(n) \text{ is true for all } n \in \mathbb{P}.
$$

b) $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n + 1)} = \frac{n}{n + 1}$; that is, $p(n) = \sum_{i=1}^{n} \frac{1}{i(i + 1)} = \frac{n}{n + 1}$

Only First Principle is needed: (B): $\frac{1}{1 \cdot 2} = \frac{1}{1 + 1}$ so $p(1)$ is true.

(I): Given $p(k - 1)$; that is, $\sum_{i=1}^{k-1} \frac{1}{i(i + 1)} = \frac{k - 1}{k(k + 1)}$, then $\sum_{i=1}^{k} \frac{1}{i(i + 1)} = \left[\frac{k - 1}{k(k + 1)}\right] + \frac{1}{k(k + 1)}$, which equals

$$
\left[\frac{(k - 1)(k + 1)}{k(k + 1)}\right] + \frac{1}{k(k + 1)} = \frac{k^2 - 1}{k(k + 1)} + \frac{1}{k(k + 1)} = \frac{k}{k + 1} \text{ so } p(k - 1) \text{ implies } p(k). \text{ By (B) & (I), } p(n) \text{ is true for all } n \in \mathbb{P}.
$$

3. Prove $p(n): n^5 - n$ is divisible by 10 for all $n \in \mathbb{P}$. [In Example 1, starting on page 137, the author relied on two lemmas: “$n^5 - n$ is divisible by 2.” and “$n^5 - n$ is divisible by 5.”]

An alternate proof proceeds as follows: (B) $1^5 - 1 = 0 = 10\cdot0$, so $p(1)$ is true.

(I): If $k^5 - k = 10q$, then $(k+1)^5 - (k+1) = k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1$

which equals $[k^5 - k] + 5k^4 + 10k^3 + 10k^2 + 5k$ which equals $10[q + 10k^3 + 10k^2 + 5k + 1]$ which equals $10[q + 10k^3 + 10k^2 + 5k + 1]$ which equals $10[q + 10k^3 + 10k^2 + 5k + 1]$ which equals $10[q + 10k^3 + 10k^2 + 5k + 1]$ which is a multiple of 10 because either $k$ or $k+1$ is even. Hence $p(k)$ implies $p(k+1)$. (B) & (I) $\Rightarrow p(n)$.

4. a) Define a sequence $(b_n)$ by: $b_0 = b_1 = 1$; $b_2 = 2b_{n-1} + b_{n-2}$ for $n \geq 2$.

$b_0 = 2b_5 + b_4 = 2[2b_4 + b_3] + b_4 = 5b_4 + 2b_3 = 5[2b_3 + b_2] + 2b_3 = 12b_3 + 5b_2 = 12[2b_2 + b_1] + 5b_2 = 29b_2 + 12b_1 = 29[2b_1 + b_0] + 12b_1 = 70b_1 + 29b_0 = 70 + 29 = 99.$

b) Define a sequence $(a_n)$ by $a_0 = a_1 = a_2 = 1$; $a_{n-2} + a_{n-3}$ for $n \geq 3$. $a_0 = a_2 + a_3 = a_4 + [a_5 + a_4] + a_6 = [a_4 + a_3] + a_5 + a_4 = [a_3 + a_2] + 2a_4 + a_3 = 2[a_2 + a_1] + 2a_3 + a_2 = 2[a_1 + a_0] + 3a_2 + 2a_1 = 9.$
5. Is the First Principle of Mathematical Induction adequate to prove the fact in Exercise 11(b) on page 234 [that all the terms are odd in the sequence \( a_n \): defined by: \( a_0 = a_1 = 1; a_n = a_{n-1} + 2a_{n-2} \) for \( n \geq 2 \)]?
Yes, because only the oddness of \( a_{n-1} \) matters in establishing the oddness of \( a_n \), as \( 2a_{n-2} \) is always even.

6. Recursively define \( a_0 = 1, a_1 = 2; a_n = \frac{(a_{n-1})^2}{a_{n-2}} \) for \( n \geq 2 \).

(a) Calculate the first few terms of the sequence: 1, 2, \( \frac{2^2}{1} = 4, \frac{4^2}{2} = 8, \frac{8^2}{4} = 16, \cdots \)

(b) Guess a general formula for \( a_n \). It appears that \( p(n): a_n = 2^n \) for \( n \geq 0 \).

(c) Prove the guess in part b). We use the Second Principle of Mathematical Induction:
(B) \( 2^0 = 1 \) and \( 2^1 = 2 = a_1 \) so Basis is established.

(I) If \( a_r = 2^r \) for all \( r \) such that \( 0 \leq r < k \), then \( a_k = \frac{(2^{k-1})^2}{2^{k-2}} = 2^{2k-2-k+2} = 2^k \) for \( k \geq 2 \).

(B) & (I) implies \( p(n) \) for \( n \geq 0 \).

7. Recursively define \( a_0 = a_1 = 1; a_n = \frac{(a_{n-1})^2 + a_{n-2}}{a_{n-1} + a_{n-2}} \) for \( n \geq 2 \).

(a) Calculate the first few terms of the sequence: 1, 1, \( \frac{12 + 1}{1 + 1} = 1, \frac{12 + 1}{1 + 1} = 1, \cdots \)

(b) Guess a general formula for \( a_n \). It appears that \( p(n): a_n = 1 \) for \( n \geq 0 \).

(c) Prove the guess in part b). We use the Second Principle of Mathematical Induction:
(B) is clearly established by definition.

(I) If \( a_r = 1 \) for all \( r \) such that \( 0 \leq r < k \), then \( a_k = \frac{12 + 1}{1 + 1} = 1 \) for \( k \geq 2 \).

(B) & (I) implies \( p(n) \) for \( n \geq 0 \).

8. Recursively define \( a_0 = 1, a_1 = 2; a_n = \frac{(a_{n-1})^2 - 1}{a_{n-2}} \) for \( n \geq 2 \).

(a) Calculate the first few terms of the sequence: 1, 2, \( \frac{2^2 - 1}{1} = 3, \frac{3^2 - 1}{2} = 4, \frac{4^2 - 1}{3} = 5, \cdots \)

(b) Using part (a), guess a general formula for \( a_n \). It seems \( a_n = n + 1 \) for \( n \in \mathbb{N} \).

(c) Use the second principle of induction to prove \( p(n): a_n = n + 1 \) for \( n \in \mathbb{N} \).
(B): Need to check first two cases: \( 0 + 1 = a_0 \) and \( 1 + 1 = a_1 \), so basis is established.

(I): If \( a_r = r + 1 \) for all \( r \) such that \( 0 \leq r < k \), then \( a_k = \frac{(a_{k-1})^2 - 1}{a_{k-2}} = \frac{((k-1)+1)^2 - 1}{(k-2)+1} = \frac{k^2 - 1}{k-1} = k + 1 \) for \( k \geq 2 \).

(B) & (I) implies \( p(n) \) for \( n \geq 0 \).

9. Recursively define \( a_0 = 0, a_1 = 1; a_n = \frac{1}{4}(a_{n-1} - a_{n-2} + 3)^2 \) for \( n \geq 2 \).

(a) Calculate the first few terms of the sequence: 0, 1, \( \frac{1}{4}(1 - 0 + 3)^2 = 4, \frac{1}{4}(4 - 1 + 3)^2 = 9, \cdots \)

(b) Using part (a), guess a general formula for \( a_n \). It seems \( a_n = n^2 \) for \( n \in \mathbb{N} \).

(c) Use the second principle of induction to prove \( p(n): a_n = n^2 \) for \( n \in \mathbb{N} \).
(B): Need to check first two cases: \( 0^2 = 0 = a_0 \) and \( 1^2 = 1 = a_1 \), so basis is established.

(I): If \( a_r = r^2 \) for all \( r \) such that \( 0 \leq r < k \), then \( a_k = \frac{1}{4}(a_{k-1} - a_{k-2} + 3)^2 = \frac{1}{4}((k-1)^2 - (k-2)^2 + 3)^2 \)
which = \( \frac{1}{4}[(k^2 - 2k + 1 + 1 - k^2 + 4k - 4 + 3)^2 = \frac{1}{4}[2k]^2 = k^2 \) for \( k \geq 2 \). That is, \( a_k = k^2 \) for \( k \geq 2 \).

(B) & (I) implies \( p(n) \) for \( n \geq 0 \).
10. Recursively define \( a_0 = 1, a_1 = 2, a_2 = 3; a_n = a_{n-2} + 2a_{n-3} \) for \( n \geq 3 \).
   a) Calculate \( a_n \) for \( n = 3, 4, 5, 6, 7 \). \( a_3 = 2 + 2(1) = 4, a_4 = 3 + 2(2) = 7, a_5 = 4 + 2(3) = 10, a_6 = 7 + 2(4) = 15, a_7 = 10 + 2(7) = 24 \).
   b) Prove that \( p(n) \): \( a_n > \left( \frac{3}{2} \right)^n \) for all \( n \geq 1 \).

   (B): Need to check 3 cases: \( a_1 = 2 > \left( \frac{3}{2} \right)^1 = \left( \frac{3}{2} \right)^1 \) and \( a_2 = 3 > \left( \frac{3}{2} \right)^2 = \left( \frac{3}{2} \right)^2 \) and \( a_3 = 4 > \left( \frac{27}{8} \right) = \left( \frac{3}{2} \right)^3 \).

   (I): If \( a_r > \left( \frac{3}{2} \right)^r \) for all \( r \) such that \( 1 \leq r < k \), where \( k \geq 4 \), then \( a_k = a_{k-2} + 2a_{k-3} + \frac{3^k}{2} = \frac{3^k}{2} + 2 \left( \frac{3}{2} \right)^{k-3} \) which equals \( \left( \frac{3}{2} \right)^{k-3} + 2 = \left( \frac{3}{2} \right)^{k-2} \left( \frac{3}{2} \right) = \left( \frac{3}{2} \right)^k \) for \( k \geq 4 \).

   (B) \& (I) implies \( p(n) \) for \( n \geq 1 \).

11. Recursively define \( a_0 = a_1 = a_2 = 1; a_n = a_{n-1} + a_{n-2} + a_{n-3} \) for \( n \geq 3 \).
   a) Calculate the first few terms of the sequence: 1, 1, 1, 1+1+1=3, 3+1+1=5, 5+3+1=9, 9+5+3=17, ...
   b) Prove that all the \( a_n \)'s are odd. [By second principle of induction]

   (B): first three terms are odd;
   (I): If \( a_r \) is odd for all \( r \) such that \( 0 \leq r < k \), where \( k \geq 3 \), then \( a_k \) is the sum of three odd numbers and is also odd.

   c) Prove that \( p(n) \): \( a_n < 2^{n-1} \) for all \( n \geq 1 \). [By second principle of induction]

   (B): Need to check 3 cases: \( a_1 = 1 \leq 2^0 = 1 \), \( a_2 = 1 \leq 2^1 = 2 \), \( a_3 = 3 \leq 2^2 = 4 \);

   (I): If \( a_r \leq 2^{r-1} \) for all \( r \) such that \( 1 \leq r < k \), where \( k \geq 4 \), then \( a_k = a_{k-2} + 2a_{k-3} \leq 2^{k-2} + 2 \left( \frac{3}{2} \right)^{k-3} \) which is \( \leq 2^{k-1} + 2 \left( \frac{3}{2} \right)^{k-2} + 2 \left( \frac{3}{2} \right)^{k-3} = 2^{k-1} \). So \( a_k \leq 2^{k-1} \) for \( k \geq 4 \).

   (B) \& (I) implies \( p(n) \) for \( n \geq 1 \).

12. Recursively define \( a_0 = 1, a_1 = 3, a_2 = 5; a_n = 3a_{n-2} + 2a_{n-3} \) for \( n \geq 3 \).
   a) Calculate \( a_n \) for \( n = 3, 4, 5, 6, 7 \). \( a_3 = 3(3) + 2(1) = 11, a_4 = 3(5) + 2(3) = 21, a_5 = 3(11) + 2(5) = 43, a_6 = 3(21) + 2(11) = 85, a_7 = 3(43) + 2(21) = 171 \).
   b) Prove that \( p(n) \): \( a_n > 2^n \) for \( n \geq 1 \). [By second principle of induction]

   (B): We need to check three cases: \( a_1 = 3 > 2^1 = 2 \), \( a_2 = 5 > 2^2 = 4 \), \( a_3 = 11 > 8 = 2^3 \);

   (I): If \( a_r > 2^r \) for all \( r \) such that \( 1 \leq r < k \), where \( k \geq 4 \), then \( a_k = 3a_{k-2} + 2a_{k-3} > 3(2^{k-2}) + 2(2^{k-3}) \) which is \( < 3(2^{k-2} + 2) = 2^{k-1} \). So \( a_k > 2^k \).

   (B) \& (I) implies \( p(n) \) for \( n \geq 1 \).

   c) Prove that \( p(n) \): \( a_n < 2^{n+1} \) for \( n \geq 1 \). [By second principle of induction]

   (B): We need to check three cases: \( a_1 = 3 < 4 = 2^1+1 \), \( a_2 = 5 < 8 = 2^2+1 \), \( a_3 = 11 < 16 = 2^{3+1} \);

   (I): If \( a_r < 2^{r+1} \) for all \( r \) such that \( 1 \leq r < k \), where \( k \geq 4 \), then \( a_k = 3a_{k-2} + 2a_{k-3} \) which is \( < 3(2^{k-2}+2) = 2^{k-1} \). So \( a_k < 2^{k+1} \).

   (B) \& (I) implies \( p(n) \) for \( n \geq 1 \).

d) Prove that \( p(n) \): \( a_n = 2a_{n-1} + (-1)^{n+1} \) for \( n \geq 1 \). [By second principle of induction]

   (B): We need to check three cases: \( a_1 = 3 = 2(1) + (1) = 2(a_0) + (-1)^{1+1} \), \( a_2 = 5 = 2(3) + (-1) = 2(a_1) + (-1)^{2+1} \), \( a_3 = 11 = 2(5) + (1) = 2(a_2) + (-1)^{3+1} \);

   (I): If \( a_r = 2a_{r-1} + (-1)^{r+1} \) for all \( r \) such that \( 1 \leq r < k \), where \( k \geq 4 \), then \( a_k = 3a_{k-2} + 2a_{k-3} \) which is \( < 3(2^{k-2}+1) + 2(-1)^{k+1} \). So \( a_k < 2^{k+1} \).

   (B) \& (I) implies \( p(n) \) for \( n \geq 1 \).
13. Recursively define $b_0 = b_1 = b_2 = 1$; $b_n = b_{n-1} + b_{n-3}$ for $n \geq 3$.

a) Calculate $b_n$ for $n = 3, 4, 5, 6$. $b_3 = 1 + 1 = 2$, $b_4 = 2 + 1 = 3$, $b_5 = 3 + 1 = 4$, $b_6 = 4 + 2 = 6$.

b) Show that $b_n \geq 2b_{n-2}$ for $n \geq 3$. [By second principle of induction]

(B): We need to check three cases: $b_3 = 2 \geq 2(1) = 2b_1$, $b_4 = 3 \geq 2(1) = 2b_2$, $b_5 = 4 \geq 2(2) = 2b_3$;

(I): If $b_r \geq 2b_{r-2}$ for all $r$ such that $3 \leq r < k$, where $k \geq 6$, then $b_k = b_{k-1} + b_{k-3} \geq 2b_{k-1-2} + 2b_{k-3-2}$

which $= 2[b_{k-3} + b_{k-5}] = 2b_{k-2}$. So $b_k \geq 2b_{k-2}$ for $k \geq 6$. (B) & (I) implies $p(n)$ for $n \geq 3$.

c) Prove the inequality $b_n \geq (\sqrt{2})^{n-2}$ for $n \geq 2$. [By second principle of induction]

(B): We check three cases:

$I_1$: $b_2 = 1 \geq 1 = (\sqrt{2})^{2-2}$,

$I_2$: $b_3 = 2 \geq \sqrt{2} = (\sqrt{2})^{2-2}$,

$I_3$: $b_4 = 3 \geq 2 = (\sqrt{2})^{2-2}$.

So $b_k \geq (\sqrt{2})^{k-2}$ for $k \geq 5$.

(B) & (I) implies $p(n)$ for $n \geq 2$.

14. Recursively define $b_0 = b_1 = b_2 = 1$; $b_n = b_{n-1} + b_{n-3}$ for $n \geq 3$. Show that $b_n \leq \left(\frac{3}{2}\right)^{n-1}$ for $n \geq 1$.

[By second principle of induction]

(B): Check three cases: $b_1 = 1 \leq 1 = \left(\frac{3}{2}\right)^{1-1}$, $b_2 = 1 \leq \frac{3}{2} = \left(\frac{3}{2}\right)^{2-1}$, $b_3 = 2 \leq \frac{3}{2} = \left(\frac{3}{2}\right)^{3-1}$

(I): If $b_r \leq \left(\frac{3}{2}\right)^{r-1}$ for all $r$ such that $1 \leq r < k$, where $k \geq 4$, then $b_k = b_{k-1} + b_{k-3}$ which

is $\leq \left(\frac{3}{2}\right)^{k-1} + \left(\frac{3}{2}\right)^{k-3} \leq \left(\frac{3}{2}\right)^{k-1} \left[\left(\frac{3}{2}\right)^{2} + 1\right] = \left(\frac{3}{2}\right)^{k-1} \left[\left(\frac{3}{2}\right)^{2} + 1\right] = \left(\frac{3}{2}\right)^{k-1}$. So $b_k \leq \left(\frac{3}{2}\right)^{k-1}$ for $k \geq 4$.

(B) & (I) implies $p(n)$ for $n \geq 1$. 