R&W 5.1 Basics of Counting

**Addition** is used generally to count the union of two sets, as in alternative occurrences. 
**Example:** Positive integers less than 100 divisible by 3 or by 5.
Divisible by 3: 33; divisible by 5: 19; divisible by 15: 6. $33+19−15 = 37$ divisible by 3 or 5.

**Union Rules:** Let S and T be finite sets:
a) If S and T are disjoint, that is, $S \cap T = \emptyset$, then $|S \cup T| = |S| + |T|$.
b) in general, $|S \cup T| = |S| + |T| - |S \cap T|$
c) If $S \subseteq T$, then $|T \setminus S| = |T| - |S|$

**Multiplication** is used generally to count the cross product of two sets, as in joint occurrence of two events. It can be extended to count joint occurrence of multiple independent events.
**Example:** Three-digit numerals representing odd numbers.
First digit: $|\{1..9\}| = 9$ ways; second digit: $|\{0..9\}| = 10$ ways; third digit: $|\{1,3,5,7,9\}| = 5$ ways.
$9 \times 10 \times 5 = 450$ odd numbers with 3-digits.

**Product Rules:** $|S_1 \times S_2 \times \ldots \times S_k| = |S_1| \times |S_2| \times \ldots \times |S_k|$

**Mappings** from domain to codomain: Counting all functions from set S to set T.
**Example:** If $|S| = 3$ and $|T| = 5$, there are $5^3 = 125$ such mappings.

**Power Rule:** $|T^S| = |T|^{|S|}$

**One-to-one** functions are counted by declining products.
**Example:** If $|S| = 3$ and $|T| = 5$, there are $5 \times 4 \times 3$ such mappings. 
(This is a 3-permutation from 5 elements.)

A *permutation* of a set is an ordered list of its elements. [Placement into positions]
An r-*permutation* of a set T is a sequence of r distinct elements of T.

A one-to-one mapping $\sigma: \{1, 2, 3, \ldots, r\} \to T$

The number of r-permutations is 

$$n(n-1)(n-2)...(n-r+1) = P(n, r) = \frac{n!}{(n-r)!}$$

A *combination* is an “unordered permutation”. [Selection for a collection.]

If S has n elements, $\binom{n}{r}$ is the number of r-element subsets of S.

EXAMPLE. Strings of 0's and 1's of length seven that have exactly three 1's
Here S is the set of seven positions in the string. Which three of them are 1's?
The first 1 could be any of seven positions, the second any of the other six, and the third any of five, but the order of choosing these positions does not matter, so we divide by the $3 \times 2 \times 1$ ways we could have done so: $(7 \times 6 \times 5)/(3 \times 2 \times 1) = 35$

$$\binom{n}{r} = \frac{n!}{(n-r)! \cdot r!} = \binom{n}{n-r}$$
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1. Calculate the following:
   a) \( \binom{8}{3} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56 \)
   b) \( \binom{8}{0} = \frac{8!}{0! \cdot 8!} = 1 \)
   c) \( \binom{8}{5} = \binom{8}{3} = 56 \)
   d) \( \binom{52}{20} - \binom{52}{2} - \frac{52 \cdot 51}{2 \cdot 1} - 1326 \)
   e) \( \binom{52}{32} - \frac{52 \cdot 51 \cdot 50}{3! \cdot 2! \cdot 1} - 1 \)
   f) \( \binom{52}{1} - \frac{52}{1} = 52 \)

2. a) Give an example of a counting problem whose answer is \( P(26, 10) \).
   The number of arrangements of 10 (different) elements from a set of 26 elements – as in 10-letter “words” with no repeated letters.
   b) Give an example of one whose answer is \( \binom{26}{10} \).
   The number of selections of 10 (different) elements from a set of 26 elements – as in subsets of 10 letters from an alphabet of 26 letters.

3. Give the value of the following:
   a) \( P(10, 1) = 10 \)
   b) \( P(10, 0) = 1 \)
   c) \( P(10, -2) + P(10, 17) = 0 + 0 = 0 \)
   d) \( P(100, 2) = 100 \cdot 99 = 9900 \)
   e) \( P(1000, 300) \)
   f) \( P(100, 2) = 100 \cdot 99 = 9900 \)

4. Let \( A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \) and \( B = \{2, 3, 5, 7, 11, 13, 17, 19\} \).
   a) \( |A \cup B| = |\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 17, 19\}| = 14 \)
   \( |A \cap B| = |\{2, 3, 5, 7\}| = 4 \)
   b) \( |A| = 2^{10} = 1024 \)
   \( |\varphi(A)| = 2^{10} = 1024 \)
   \( |A \circ B| = |A \cup B| - |A \cap B| = 14 - 4 = 10 \) and \( |A \circ B| = |A| + |B| - |A \cap B| = 10 + 8 - 4 = 14 \).

   c) How many 4-element subsets of \( A \) are there? \( \binom{10}{4} = \frac{10!}{6! \cdot 4!} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 210 \)
   d) How many 4-elements subsets of \( A \) consist of three even and one odd number?

From 5 even elements choose three; and from 5 odd elements choose one: \( \binom{5}{3} \cdot \binom{5}{1} = 10 \cdot 5 = 50 \)

5. Given 20 different types of inputs to a program, in how many ways can 8 of them be selected if
   a) order does not matter: \( \binom{20}{8} = \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13}{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 125970 \)
   b) order does matter: \( P(20, 8) = 125970 \cdot 8! = 5079110400 \)

6. A certain class consists of 12 men and 16 women. How many committees can be chosen from this class consisting of
   a) seven people: \( \binom{28}{7} = 1184040 \)
   b) three men and four women: \( \binom{12}{3} \cdot \binom{16}{4} = (220)(1820) = 400400 \)
   c) seven women or seven men: 0M&7W or 7M&0W: \( \binom{16}{7} + \binom{12}{7} = 11440 + 792 = 12232 \)
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7. a) How many committees consisting of 4 people can be chosen from 9 people? \( \binom{9}{4} = 126 \)
b) Redo part a) if there are two people, Ann and Bob, who will not serve on the same committee.
   **We omit those committees that would have Ann, Bob, and 2 others:** \( 126 - \left( \frac{1}{4} \right) \cdot \left( \frac{1}{4} \right) \cdot \left( \frac{2}{2} \right) = 105 \)

8. How many committees consisting of 4 men and 4 women can be chosen from a group of 8 men and 6 women? \( \left( \frac{8}{4} \right) \cdot \left( \frac{6}{4} \right) = (70) \cdot (15) = 1050 \)

9. Let S = \{ a, b, c, d \} and T = \{ 1, 2, 3, 4, 5, 6, 7 \}
a) How many one-to-one functions are there from T into S? **None, because |T| > |S|.**
b) How many one-to-one functions are there from S into T? \( 7 \cdot 6 \cdot 5 \cdot 4 = 840 = P(7, 4) \).
c) How many functions are there from S into T? \( 7^4 = 2401 \).

10. Let P = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9 \} and Q = \{ A, B, C, D, E \}
a) How many 4-element subsets of P are there? \( \binom{9}{4} = 126 \)
b) How many permutations, that is 5-permutations, of Q are there? \( P(5, 5) = 5! = 120 \).
c) How many license plates are there consisting of three letters from Q followed by two numbers from P? [Repetition is allowed; for example, DAD88 is allowed.] \( 5^3 \cdot 9^2 = 10125 \).

11. Cards are drawn from a deck of 52 cards with replacement.
a) In how many ways can ten cards be drawn so that the tenth card is not a repetition?
   **We may suppose the first card is different from the last nine:** \( 52 \cdot 51^9 \approx 1.21 \cdot 10^{17} \).
b) In how many ways can ten cards be drawn so that the tenth card is a repetition?
   **We subtract the number in part a) from the total ways to draw:** \( 52^{10} - 52 \cdot 51^9 \approx 2.31 \cdot 10^{16} \).

15. Count the number of poker hands of the following kinds:
a) four of a kind: \( (13 \text{kinds})(\text{choose all 4 suits})(\text{choose 1 from 48 other cards}) = 13 \cdot 1 \cdot 48 = 624 \)
b) ordinary flush [not straight or royal]: \( (4 \text{suits})(\text{choose 5 from 13 in suit}) - [36+4] = 5108 \)
c) three of a kind: \( (13 \text{kinds})(\text{choose 3 of 4 suits})(\text{choose 2 of 12 other kinds})(4\text{suits})(4\text{suits}) = 13 \cdot 4 \cdot 66 \cdot 4 \cdot 4 = 54912 \).
d) one pair: \( (13 \text{kinds})(\text{choose 2 of 4 suits})(\text{choose 3 of 12 other kinds})(4\text{suits})(4\text{suits})(4\text{suits}) = 13 \cdot 6 \cdot 220 \cdot 4 \cdot 4 \cdot 4 = 1098240 \).