An **ordered partition** of a set is a sequence of pairwise disjoint nonempty subsets for which the union of these subsets is the set itself.

**Example** Members of a group of 15 people are assigned to three committees of size 3, 4, 5 with no person serving on more than one committee. By considering the 3 people who are on none of the committees, we have a partition of the group.

By using the product rule, and counting the ways of populating each committee using combinations, we see that there are \[ \binom{15}{3} \cdot \binom{12}{4} \cdot \binom{8}{5} \cdot \binom{3}{3} \] ways to partition this group.

Alternatively we can consider the number of permutations representing the order in which people are assigned (or not) to a committee, and reducing that number by the number of ways each committee could have been populated. By this means we see that there are \[ \frac{15!}{3!4!5!3!} \] ways to partition this group.

Simplifying the product of the combinations above confirms this formula.

The formula may be represented as a multinomial coefficient \( \binom{15}{3, 4, 5, 3} \) where this number represents the coefficient of the term in the expansion of \( (a+b+c+d)^{15} \) which has the form \( a^3b^4c^5d^3 \).

A **multiset** is a collection of elements of identifiable types. For example, a collection of fruit that has 3 apples, 4 oranges, 5 pears, and 3 bananas is different from a collection of 7 apples and 8 bananas.

Putting index (position) labels on the elements allows us to treat a multiset as an ordinary set. The former multiset might be labeled as \{a_{1}, a_{2}, a_{3}, o_{1}, o_{2}, ..., o_{4}, p_{1}, p_{2}, ..., p_{5}, b_{1}, b_{2}, b_{3} \}.

A combination of elements of a multiset is just a subset of the elements, noting only how many of each type. A permutation of elements of a multiset is an ordered listing of the elements of the multiset.

**EXAMPLE** Tiles with letters can be arranged in a line to form strings called words.
For example, if letter A is on 3 tiles, and letter B is on 1 tile, and letter N is on 2 tiles, the word BANANA can be spelled. Of course other strings such as ANNABA are also possible.

Because a string looks the same no matter which of the 3 A’s, for example, is placed first, we count the number of distinguishable strings by regarding this as an ordered partition: \[ \frac{6!}{3!1!2!} \]

An **unordered partition** results when the collection of pairwise disjoint nonempty subsets for which the union of these subsets is the set itself is not in the form of a sequence. That is, the subsets may only be distinguished by size, not by contents or labels.

**EXAMPLE** Six students are grouped in pairs, to work on a set of problems.

a) If each pair works on a different problem, there are \[ \frac{6!}{2!2!2!} = 90 \] ways.

b) If each pair works on the same problems, there are only \[ \frac{90}{3!} = 30 \] ways.
1. From a total of 15 people, 3 committees consisting of 3, 4, and 5 people respectively are to be chosen.
a) How many such sets of committees are possible if no person may serve on more than one committee?
This is an ordered partition: 3-4-5-3: \( \binom{15}{3} \cdot \binom{12}{4} \cdot \binom{8}{5} \cdot \binom{3}{3} = 455 \cdot 495 \cdot 56 \cdot 1 = 12612600. \)
b) How many such sets of committees are possible if there is no restriction on the number of committees on which a person may serve? Not a partition: \( \binom{15}{3} \cdot \binom{15}{4} \cdot \binom{15}{5} = 455 \cdot 1365 \cdot 3003. \)

2. Compare the following:

a) \( \binom{7}{2} \cdot \binom{5}{2} = \frac{7!}{2!5!} \cdot \frac{5!}{2!3!} = \frac{7!}{2!1!3!} = \frac{7!}{2!1!3!} \)

b) \( \binom{12}{3} \cdot \binom{9}{4} = \frac{12!}{3!9!} \cdot \frac{9!}{4!5!} = \frac{12!}{3!4!5!} \)

c) \( \binom{n}{k} \cdot \binom{n-k}{r} = \frac{n!}{k!(n-k)!} \cdot \frac{(n-k)!}{r!(n-k-r)!} = \frac{n!}{k!r!(n-k-r)!} \)

d) \( \binom{n}{r+s} = \frac{n!}{r!s!} = \binom{n}{r}, \) for \( r+s = n, \)

e) \( \binom{n}{1...1} = \frac{n!}{1!...1!} = n! \)

3. Three pairwise disjoint teams are to be selected from a group of thirteen students, with all three teams competing in the same programming contest. In how many ways can the teams be formed if they are to have

a) 5, 3, and 2 students? Ordered partitions: \( \binom{13}{5} \cdot \binom{5}{3} \cdot \binom{2}{2} = \frac{13!}{5!3!2!} = 720720. \)
b) 4, 3, and 3 students? Ordered partitions; 2 teams equivalent: \( \frac{13!}{4!3!3!} \cdot \frac{1}{2!} = 600600. \)
c) 3 students each? Ordered partitions; 3 teams equivalent: \( \frac{13!}{3!3!4!} \cdot \frac{1}{3!} = 200200 \)

4. How many different signals can be created by lining up nine flags in a vertical column if 3 flags are white, 2 are red, and 4 are blue? Ordered partitions: \( \frac{9!}{3!2!4!} = 1260. \)
5. Let \( S \) be the set of all sequences of 0’s, 1’s, and 2’s of length ten. For example, \( S \) contains 0 2 1 1 0 1 2 2 0 1.

a) How many elements are in \( S \)? By product or power rule: \( 3^{10} \) which equals 59049.

b) How many sequences in \( S \) have exactly five 0’s and five 1’s? Count where 0’s are: \( \binom{10}{5} = 252 \)

c) How many sequences in \( S \) have exactly three 0’s and seven 1’s? Count where 0’s are:

d) How many sequences in \( S \) have exactly three 0’s?

Count where 0’s are and fill other slots with 1 or 2: \( \binom{10}{3} \cdot 2^7 = 15360 \)

e) How many sequences in \( S \) have exactly three 0’s, four 1’s, three 2’s?

Ordered partitions: \( \frac{10!}{3!4!3!} = 4200 \)

f) How many sequences in \( S \) have at least one 0, at least one 1, and at least one 2?

Use inclusion–exclusion on sequences lacking one or more of these digits:
\[ 3^{10} - [2^{10} + 2^{10} + 2^{10} - 1^{10} - 1^{10} - 1^{10} + 0^{10}] = 55980. \]

6. Find the number of permutations that can be formed from all the letters of the following words.

a) FLORIDA \( 7! = 5040 \)

b) CALIFORNIA \( \frac{10!}{2!2!} = 907200. \)

c) MISSISSIPPI \( \frac{11!}{4!4!2!} = 34650. \)

d) OHIO \( \frac{4!}{2!} = 12 \)

7. a) How many 4-digit numbers can be formed using only the digits 3, 4, 5, 6, and 7? \( 5^4 = 625. \)

b) How many of the numbers in part (a) have at least one digit appear more than once?

Subtract those with no digits repeated: \( 5^4 - 5 \cdot 4 \cdot 3 \cdot 2 = 505. \)

c) How many of the numbers in part (a) are even? [Last digit is 4 or 6] \( 5^3 \cdot 2 = 250. \)

d) How many of the numbers in part (b) are bigger than 5000? [First digit not 3 or 4] \( 3 \cdot 5^3 = 375 \) and at least on digit appears more than once: \( 375 - 3 \cdot 4 \cdot 3 \cdot 2 = 303. \)