6.3 Trees
A path in an undirected graph $G = (V, E)$ is a sequence of edges that connect adjacent vertices. An undirected graph $G = (V, E)$ is **connected** if there is a path between any pair of vertices. In a simple graph this path can be denoted by the sequence of vertices, because there are no multiple edges connecting vertices.

A **circuit** is a path with one or more edges in which the last vertex in the sequence is the same as the first vertex.

A path or circuit is called **simple** if it does not contain the same edge more than once.

**Theorem** There is a simple path between any pair of distinct vertices of a connected undirected graph.

A **tree** is a connected acyclic graph, commonly used in data structures. Trees have no loops and no parallel edges.

There are two non-isomorphic trees with four vertices.

There is a simple path between any pair of distinct vertices of a connected undirected graph.

**Theorem 1** Let $e$ be an edge of a connected graph $G$. Then the following are equivalent:

a) $G \setminus \{e\}$ is connected.

b) $e$ is an edge of some cycle in $G$.

c) $e$ is an edge of some simple closed path in $G$.

A minimal subgraph that connects all the vertices of a graph is called a **spanning tree**.

**Theorem 2** Every finite connected graph $G$ has a spanning tree.

An undirected graph is a tree iff there is a unique simple path between any pair of its vertices.

**Theorem 3** Let $G$ be a graph with more than one vertex, no loops, and no parallel edges. Then the following are equivalent:

a) $G$ is a tree.

b) Each pair of distinct vertices is connected by exactly one simple path.

c) $G$ is connected, but will not be if any edge is removed.

d) $G$ is acyclic, but will not be if any edge is added.

Vertices of degree one are called **leaves**. [Singular is leaf.]

**Lemma 1** A finite tree with at least one edge has at least two leaves.

**Lemma 2** A tree with $n$ vertices has exactly $n - 1$ edges.

**Theorem 4** Let $G$ be a finite graph with $n$ vertices, no loops, and no parallel edges. Then the following are equivalent:

a) $G$ is a tree.

b) $G$ is acyclic and has $n - 1$ edges.

c) $G$ is connected and has $n - 1$ edges.

A **rooted tree** is a tree in which one vertex is distinguished as the root and every edge is directed away from the root.
Note that different choices of the root produce different rooted trees.

A rooted tree in which each internal vertex [which is an ancestor of its descendents] has no more than two children [immediate descendents] is called a **binary tree**.

**Example** Seven parallel processors may be connected in a full binary tree to efficiently add eight numbers in three cycles; seven cycles would be required for one processor to add these numbers serially.

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P1: (x1 + x2); P5: (x3 + x4); P6: (x5 + x6); P7: (x7 + x8)
P2: [(x1 + x2) + (x3 + x4)]; P3: [(x5 + x6) + (x7 + x8)]
P4: (x1 + x2); P6: (x5 + x6); P7: (x7 + x8)
P1: {[(x1 + x2) + (x3 + x4)] + [(x5 + x6) + (x7 + x8)]}
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Exercises p 243. 1 – 6