Predicates and Quantifiers

Sentences that assert a property about one or more variables are neither true nor false unless values of the variables are specified.

Example: $2^n = n^2$ is true for natural numbers 2 and 4, false otherwise.

For ordinary multiplication of real numbers, $x \cdot y = y \cdot x$, but may be false if $x$ and $y$ are matrices.

The variables in such sentences are the subjects and the property described is the predicate. The sentence “$x > 3$” can be denoted by the statement $P(x)$, which is a propositional function. $P(4)$ is true, but $P(2)$ is false.

Note: If ... then ... in programming is not the same as in logic.

The expression following if is typically a predicate, say $C(k) : k > 0$

and the expression following then is typically an executable statement, say $k := k – 1$

When the predicate is true the executable statement is performed; otherwise it is skipped.

Quantification is another way to create a proposition from a propositional function. In the following, quantifiers are treated informally.

Universal: for all $\forall x \in \mathbb{R}$ \( P(x) : x^2 = x \) is False. Only true when $x$ is in \{0, 1\}

Existential: for some $\exists n \in \mathbb{N}$, $2^n = n^2$ is True. Because $n$ could be in \{2, 4\}

Other examples:

S(x) : $\exists x \in \mathbb{R}, x^2 = x$ is True. \{0, 1\}

T(x, y) : $x^2 = y^2$ implies $x = y$, $\forall x, y$ is True on the real interval $[0, \infty)$ or on the real interval $(- \infty, 0]$.

C(x, y, z) : $xy = xz$ implies $y = z$, $\forall x, y, z \in \mathbb{R}$ is False (x may be zero)

L(n) : $n^3 < 3^n$ for all $n \in \mathbb{N}$ is not true because $3^3 \not< 3^3$.

Counterexamples and instances:

Finding one instance of a value for which $P(x)$ is true, shows that “$\exists x, P(x)$” is true.

Finding one instance of a value for which $P(x)$ is false, shows that “$\forall x, P(x)$” is false.

“(x+1)^2 \geq x^2, \forall x \in \mathbb{R}” is false because $(2 + 1)^2 = 9 \not\geq 4 = (-2)^2$.

There are no counterexamples for $x \geq -\frac{1}{2}$.

Negations

$\neg \forall x P(x) \equiv \exists x \neg P(x)$ “Not every student in Discrete Math has completed Calculus” is equivalent to “Some student in Discrete Math has not completed Calculus.”

$\neg \exists x Q(x) \equiv \forall x \neg Q(x)$ “No math majors are required to take Discrete Math” is equivalent to “Every math major is not required to take Discrete Math”.

Ambiguities in natural language often make easy application of these forms challenging.

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