Functions

A function \( f \) assigns to each element \( x \) in some set \( S \) a single element in a set \( T \). 

\( f \) is defined on \( S \) with values in \( T \).

We write \( f : S \rightarrow T \) and read \( f \) maps \( S \) into \( T \).

The set \( S \) is called the **domain** of \( f \), \( \text{Dom}(f) \)

The set \( T \) is called a **codomain** of \( f \)

\( f(x) \) is called the **image** of \( x \) under \( f \); the set of all images is the **range** of \( f \), \( \text{Ran}(f) \)

If \( C \) is a subset of \( S \), then the set of all images of elements in \( C \) is the **image** of \( C \), \( f(C) \).

For \( C \) a subset of \( S \) and \( f : S \rightarrow T \), \( f(C) \) is a subset of \( \text{Ran}(f) \) which is a subset of the codomain.

**Examples:** \( g(n) = n^3 - 73n + 5 \quad g : \mathbb{N} \rightarrow \mathbb{Z} \)

absolute value: \( |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \quad F(x) = |x|, \text{ maps } \mathbb{R} \text{ to } [0, \infty) \)

floor function: \( \lfloor x \rfloor \text{ maps } \mathbb{R} \text{ to } \mathbb{Z} \).

\[
f(m, n) = \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{m-1}{2} \right\rfloor
\]

The **graph** of \( f : S \rightarrow T \) is \( \{(x, y) \in S \times T : y = f(x)\} \)

Then we get the **formal definition:** A function with domain \( S \) and codomain \( T \) is a subset \( G \) of \( S \times T \) for which for each \( x \in S \) there is exactly one \( y \in T \) such that \( (x, y) \in G \).

**One-to-one and onto**

Class size limit is 25; each student registers in next available (numbered) slot. \( F:S \rightarrow \mathbb{N} \)

Each registrant has only one slot and each slot is filled in at most one way.

If each **filled slot** has only one student, \( F \) is a **one-to-one** map. \( x_1, x_2 \in S, x_1 \neq x_2 \Rightarrow F(x_1) \neq F(x_2) \), or use contrapositive. For any \( y \) in \( \mathbb{N} \) there is at most one \( x \) in \( X \) in \( S \) such that \( (x, y) \) is in \( G \)

If every slot is filled, \( F \) maps \( S \) **onto** \( \mathbb{N} \).

\( \text{Im}(F) = T \) For any \( y \) in \( T \), there is at least one \( x \) in \( S \) such that \( (x, y) \) is in \( G \)

**Examples**

\( f(n) = 2n \) is a one-to-one map from \( \mathbb{N} \) to \( \mathbb{N} \), but is not onto.

\( G(x) = x^2 \) on \( \{-1, 0, 1\} \) to \( \{0, 1\} \) is not one-to-one but is onto.

\( G \) mapping \( \mathbb{R^+} \) to \( \mathbb{R^+} \) is onto and one-to-one

\( G \) mapping \( \mathbb{R^+} \) to \( \mathbb{R} \) is not onto \( \mathbb{R} \) but is one-to-one

\( \text{length}(w) \) is onto \( \mathbb{N} \), but is not one-to-one (unless \( \Sigma \) has only one letter).

A function that is one-to-one **and** onto is called a **one-to-one correspondence**.