MAT251 Proof by mathematical induction
Whenever it is desired to prove that an infinite sequence of statements contains only true statements, it is usually appropriate to use the method of mathematical induction. This method is a deductive (direct) method that has only two steps: BASIS and IMPLICATION. Unless both steps can be accomplished, the theorem is not proved.

To show P(n) [a statement form that depends on variable n], do:
(1) BASIS: Establish P(m) is true for some initial value of m [This is usually easy.]
(2) IMPLICATION: Prove P(k) \( \Rightarrow P(k+1) \) for any \( k \geq m \). [This is often not easy.]

Then by induction, we know P(n) is true for every \( n \geq m \).

Principle of mathematical induction The set \( \{ n \in \mathbb{P} : P(n) \text{ is true} \} = P \) provided
(B) P(1) is true and (I) P(k+1) is true whenever P(k) is true.

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Theorem A For any natural number \( n \geq 1 \), the value \( (n^3 - n) \) is divisible by 3.
(B) Clearly \( 1^3 - 1 = 0 \) which is a multiple of 3.
(I) Consider \( (k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1 \) which \( = k^3 - k + 3k^2 + 3k \)
Now if \( k^3 - k \) is divisible by 3, then \( (k+1)^3 - (k+1) = 3q + 3k^2 + 3k = 3r \) for some \( r \);
that is, \( (k+1)^3 - (k+1) \) is divisible by 3.
Then by induction we know the theorem is true for all \( n \geq 1 \).

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Theorem B If a set S has \( n \) elements, then the power set, \( \mathcal{P}(S) \), has \( 2^n \) elements, where \( n \geq 1 \).
(B) Clearly a set S with only one element, \( S = \{ a \} \), has 2 subsets: \( \emptyset \) and \( S \); that is, \( |\mathcal{P}(S)| = 2^1 \).
(I) Given any set S with \( (k+1) \) elements, for some \( k \geq 1 \), we may represent by \( S_k \) the subset containing only the first \( k \) elements. Then S is the union of \( S_k \) with the subset containing only the last element;
that is, \( S = \{ a_1, a_2, a_3, ..., a_k \} \cup \{ a_{k+1} \} \).
Now if \( |\mathcal{P}(S_k)| = 2^k \), where the element \( a_{k+1} \) is not contained in any of those subsets, then there are \( 2^k \) additional subsets formed by adjoining this element to each of those.
In total, there are \( 2^k + 2^k = 2^k(1+1) = 2^k(2) = 2^{k+1} \) subsets. That is, \( |\mathcal{P}(S)| = 2^{k+1} \).
Then by induction we know the theorem is true for all \( n \geq 1 \).

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Theorem C The sum of the first \( n \) cubes equals the square of half of \( n(n+1) \). \( \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} \)
(B) Clearly, \( 1^3 = 1^2(1+1)^2/4 \).
(I) Consider \( \sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3 \), by definition of sum. Now if \( \sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4} \),
then \( \sum_{i=1}^{k+1} i^3 = \frac{k^2(k+1)^2}{4} + (k+1)(k+1)^2 = (k+1)^2 \left( \frac{k^2}{4} + \frac{4(k+1)}{4} \right) = \frac{(k+1)^2(2k+2)^2}{4} \).
Then by induction we know the theorem is true for all \( n \geq 1 \).

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Theorem D \( 2^a > 1 + 3n \) for \( n \geq 4 \).
(B) Clearly \( 2^4 = 16 > 13 = 1 + 3(4) \).
(I) If \( 2^k > 1 + 3k \) for some \( k \geq 4 \), then \( 2^{k+1} = 2^k(2) = 2^k + 2^k > (1 + 3k) + (1 + 3k) \) which \( \geq 1 + 3k + 13 \) (if \( k \geq 4 \))
which \( > 1 + 3k + 3 = 1 + 3(k+1) \); that is, \( 2^{k+1} > 1 + 3(k+1) \).
Then by induction we know the theorem is true for all \( n \geq 4 \).