Proof by mathematical induction using a strong hypothesis

Occasionally a proof by mathematical induction is made easier by using a strong hypothesis:
To show P(n) [a statement form that depends on variable n], do:
1) BASIS: Establish \[P(m)\land P(m+1)\land\ldots\land P(m+r)\] is true for some initial value of m
2) IMPLICATION: Prove \[P(m)\land P(m+1)\land\ldots\land P(m+r)\land\ldots\land P(k)\implies P(k+1)\] for any \(k \geq m+r\)

Then by strong induction, we know P(n) is true for every \(n \geq m\).

Note that \(r = 0\) gives the simple principle of induction. Also it can be shown that the principle of strong induction follows from simple induction.

Example:
Consider a sequence defined by \(a_0 = 1, a_1 = 2, a_2 = 3\); and \(a_n = a_{(n-2)} + 2a_{(n-3)}\) for \(n \geq 3\).
We compute the next five values as follows:
\[a_3 = 2 + 2(1) = 4, a_4 = 3 + 2(2) = 7, a_5 = 4 + 2(3) = 10, a_6 = 7 + 2(4) = 15, a_7 = 10 + 2(7) = 24.\]

It appears that, after the first term, the values are increasing at a rate of about 150\%. We may establish by induction that \(P(n) : a_n > \left(\frac{3}{2}\right)^n\) for \(n \geq 1\).
Because the definition of the general term requires knowledge not just of the term immediately preceding, but of terms two and three preceding, it is useful to use a strong hypothesis for induction.

BASIS: \(a_1 = 2 > 1.5 = \left(\frac{3}{2}\right)\) so P(1) is true.
\[a_2 = 3 > 2.25 = \left(\frac{3}{2}\right)^2\] so P(2) is true
\[a_3 = 4 > 3.375 = \left(\frac{3}{2}\right)^3\] so P(3) is true.

INDUCTION: Given P(j) is true for \(1 \leq j \leq k\) where \(k \geq 3\); that is, \(a_j > \left(\frac{3}{2}\right)^j\), we establish the desired result by a sequence of substitutions and inequalities.
then \(a_{(k+1)} = a_{(k-1)} + 2(a_{(k-2)})\) and by the strong hypothesis this must be,
\[> \left(\frac{3}{2}\right)^{(k-1)} + 2\left(\frac{3}{2}\right)^{(k-2)}\] and factoring out \(\left(\frac{3}{2}\right)^{(k-2)}\) this
\[= \left(\frac{3}{2}\right)^{(k-2)}[\left(\frac{3}{2}\right) + 2]\] which
\[= \left(\frac{3}{2}\right)^{(k-2)}[3.5]\] which
\[> \left(\frac{3}{2}\right)^{(k-2)}[3.375]\] which
\[= \left(\frac{3}{2}\right)^{(k-2)}[27/8]\] which
\[= \left(\frac{3}{2}\right)^{(k+1)}\] and the result is established.

Hence by strong induction P(n) is true for all \(n \geq 1\).

Another example:
Consider the sequence defined by \(a_0 = 1, a_1 = 2;\) and \(a_n = \frac{\left(\frac{a_{(n-1)}}{a_{(n-2)}}\right)^2}{a_{(n-1)}}\) for \(n \geq 2\).
We compute the next three values as follows:
\[a_2 = 2^2/1 = 4, a_3 = 4^2/2 = 8, a_4 = 8^2/4 = 16.\]

It appears that each term is a power of 2. We may establish by induction that \(P(n) : a_n = 2^n\) for \(n \geq 0\).

BASIS: \(a_0 = 1 = 2^0\), and \(a_1 = 2 = 2^1\) so P(0) and P(1) are both true.

INDUCTION: Given P(j) is true for \(0 \leq j \leq k\) where \(k \geq 1\); that is, \(a_j = 2^j\), then
\[a_{(k+1)} = \left(\frac{a_{(k+1-1)}}{a_{(k+1-2)}}\right)^2 = \left(\frac{2^k}{2^{(k-1)}}\right)^2 = 2^{(k+1)}\] and the result is established.

Hence by strong induction P(n) is true for all \(n \geq 0\).