A (binary) relation from S to T is a subset $R$ of $S \times T$, any subset.. If $S = T$, then any subset of $S \times S$ is called a relation on $S$.

It generalizes the function mapping concept in that it need not be a set of each to one pairings. We say $s$ is $R$-related to $t$; that is, $s R t$ if and only if $(s, t) \in R$.

A function is a special kind of relation: for each $s$ in $S$ there is exactly one $t$ in $T$ such that $s R t$. That is, $f(x)$ is that unique element in $T$ such that $(x, f(x)) \in R$.

Consider each of the relations on $S = \{0, 1, 2, 3\}$. Which relations are functions?

a) $(m, n) \in R_1$ if $m + n = 3$. $R_1 = \{(0, 3), (1, 2), (2, 1), (3, 0)\}$. [This is the only function.]

b) $(m, n) \in R_2$ if $m - n$ is even. $R_2 = \{(0, 0), (0, 2), (1, 1), (1, 3), (2, 0), (2, 2), (3, 1), (3, 3)\}$.

c) $(m, n) \in R_3$ if $m \leq n$. $R_3 = \{(0, 0), (0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$.

d) $(m, n) \in R_4$ if $m + n \leq 4$. $R_4 = \{(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(1,3),(2,0),(2,1),(2,2),(3,0),(3,1)\}$

e) $(m, n) \in R_5$ if max $\{m, n\} = 3$. $R_5 = \{(0, 3), (1, 3), (2, 3), (3, 0), (3, 1), (3, 2), (3, 3)\}$

Properties of relations

(R) Reflexive property means that $(x, x)$ is in the relation for any $x$ in $S$.

(S) Symmetric property means that if $(x, y)$ is in the relation then $(y, x)$ is also.

(T) Transitive property means that if $(x, y)$ and $(y, z)$ are in the relation, then $(x, z)$ is also.

For equality, these properties say

$x = x$ for any $x$ in $S$;

if $x = y$ then $y = x$;

if $x = y$ and $y = z$, then $x = z$.

Any relation that has (R), (S), (T) properties is called an equivalence relation.

A relation is antireflexive (AR) iff $(x, x)$ is not in the relation for any $x$ in $S$.

A relation is antisymmetric (AS) iff $(x, y)$ and $(y, x)$ are both in $S$ only if $x = y$.

A relation is non-symmetric (NS) iff $(x, y)$ and $(y, x)$ are not both in $S$.

EXAMPLE Equality is the relation $\{(x, x) : x \in S\}$

Note that $=$ is (R), (S), and (T).

EXAMPLE LTE is the relation $\{(x, y) : x \leq y\}$

It is reflexive (R): $x \leq x$ for all $x \in \mathbb{R}$

It is antisymmetric (AS): $x \leq y$ and $y \leq x$ imply that $x = y$

It is transitive (T): $x \leq y$ and $y \leq z$ imply that $x \leq z$.

EXAMPLE LT is the relation $\{(x, y) : x < y\}$

It is antireflexive: (AR): $x < x$ is never true.

It is non-symmetric (NS): $x < y$ and $y < x$ is never true.

It is transitive: (T): $x < y$ and $y < z$ imply that $x < z$.

Composition of relations

If $R$ is a relation from set $A$ to set $B$ and $S$ is a relation from set $B$ to set $C$, then the composite of $R$ and $S$ is the set of pairs $(a, c)$ such that $a$ is in $A$ and $c$ is in $C$ and there is an element $b$ in $B$ such that $(a, b)$ is in $R$ and $(b, c)$ is in $S$. The composition of $R$ with $S$ is denoted $S \circ R$. 
