21. Theorem: “If \( n \) is an integer and \( n^3 + 5 \) is odd, then \( n \) is even.”

a) **Proof of the contrapositive**: If \( n \) is not even, then \( n \) equals \( 2k + 1 \) for some integer \( k \). So \( n^3 + 5 \) equals \( (8k^3 + 12k^2 + 6k + 1) + 5 \) which equals \( 2(4k^3 + 6k^2 + 3k + 3) \). So \( n^3 + 5 \) is not odd.

b) **Proof by contradiction**: Suppose \( n^3 + 5 \) is odd and \( n \) is not even. Then \( n^3 + 5 \) being odd implies \( n^3 \) must be even; but \( n \) being odd implies \( n^3 \) is odd. It is a contradiction for \( n^3 \) to be even and to be odd. Hence, \( n^3 + 5 \) being odd implies \( n \) is even.

23. Theorem: “The sum of two odd integers is even.”

**Direct proof**: \( (2k+1) + (2j + 1) = 2k + 2j + 2 = 2(k + j +1) \).

25. Theorem: “The sum of an irrational number and a rational number is irrational.”

**Proof by contradiction**: Let \( x \) be irrational, and let \( r = \frac{p}{q} \) be rational, and let \( x + r = s \). Suppose the sum \( s \) is *not* irrational, then the difference \( s - r \) which equals \( x \) must also be rational. This is a contradiction of the hypothesis that \( x \) is irrational. Hence, \( x + r \) must be irrational.

27. For the conjecture that the product of two irrational numbers is irrational, we consider the counterexample: The product of the irrational number \( \sqrt{2} \) with itself: \( \sqrt{2} \cdot \sqrt{2} \) equals 2, a rational number.

31. To provide that at least 10 of any 64 days must fall on the same day of the week, consider the contrary: “No more than 9 days fall on any day of the week.” Then at most 63 (= 9 times 7) days are accounted for, not 64. [**Proof by contradiction**.]

35. To prove the triangle inequality, \( |x| + |y| \geq |x + y| \), we consider *four cases* [quadrants in x-y plane]:

1) \( x \) and \( y \) both nonnegative: \( |x| + |y| = x + y = |x + y| \) and result is true.
2) \( x \) and \( y \) both negative: \( |x| + |y| = (-x) + (-y) = -(x + y) = |x + y| \) and result is true.
3) \( x \) negative and \( y \) nonnegative: Result is trivially true if \( y = 0 \), so let \( x < 0 \) and \( y > 0 \). Hence \( -x < |x| \) and \( -y < |y| \) and adding gives \( -(x) + (-y) < x + y < |x| + |y| \) That is, \( -|x + y| < (x + y) < |x| + |y| \) so \( |x + y| < |x| + |y| \) and result is true.
4) \( x \) nonnegative and \( y \) negative: This case is symmetric to case 3) so result is true.

39. Theorem: “For positive integer \( n \), \( n \) is odd if and only if \( 5n + 6 \) is odd.” : \( p \leftrightarrow q \)

**Direct proof of** \( p \leftarrow q \): If \( n \) is odd, then \( n = 2k + 1 \), so \( 5n + 6 = 5(2k + 1) + 6 = 10k + 11 = 2(5k + 5) + 1 \) which is odd.

**Indirect proof of** \( q \leftarrow p \), by proving \( \neg p \rightarrow \neg q \): If \( n \) is not odd, then \( n = 2k \), so \( 5n + 6 = 5(2k) + 6 = 2(5k + 3) \) which is *not* odd.

49. A constructive proof that there are 100 consecutive positive integers that are not perfect squares: Consider the perfect squares \( n^2 \) and \( (n+1)^2 \). By algebra, we see there are \( 2n \) positive integers strictly between them that are not squares. If \( n = 50 \), we see that 2501, 2502, ..., 2600 meet the criterion.

55. Theorem: “For odd integers \( a \) and \( b \), with \( a \cdot b \), there is a unique integer \( c \) such that \( |a - c| = |b - c| \).”

Either \( (a - c) = (b - c) \) or \( (a - c) = -(b - c) \). The first case implies \( a = b \), in contradiction of the given. The second case implies \( (a + b) = 2c \) and \( c \) is \( (a + b)/2 \) which is an integer because \( a \) and \( b \) are odd.