3.1 p223.

1. Prove that the product of three consecutive integers is divisible by 6.

Let the product \( P = n(n+1)(n+2) \),
then \( 2|n \) or \( 2|(n+1) \) by cases [where \( n = 2k + r \) for \( r = 0 \) or \( 1 \)], so \( 2|P \).
And \( 3|n \) or \( 3|(n+2) \) or \( 3|(n+1) \) by cases [where \( n = 3q + r \) for \( r = 0 \), \( 1 \), or \( 2 \)], so \( 3|P \).
Because \( 2|P \) and \( 3|P \) [where 2 and 3 are (relatively ) prime], then \( 2\times3 = 6|P \).
Remark: Later we will prove this theorem by mathematical induction.

3. Prove that if \( n \) is an odd positive integer, then \( n^4 \equiv 1 \pmod{16} \).

Let \( n = 2k + 1 \), then \( n^4 = 16k^4 + 32k^3 + 24k^2 + 8k + 1 \)
\[ = 8k[2k^3 + 4k^2 + 3k + 1] + 1 \]
\[ = 8k(k+1)[2k^2 + 2k + 1] + 1 \]
Now \( 2|[k(k+1)] \) so \( 16|[8k(k+1)(2k^2 + 2k + 1)] \) hence \( 8k(k+1)(2k^2 + 2k + 1) + 1 = n^4 \equiv 1 \pmod{16} \).

7. Prove there are no integers \( x \) and \( y \) such that \( 3x^2 - 8y = 1 \).

Rewrite the equation as \( 3x^2 - 1 = 8y \) and reduce both sides (mod 8):
Then \( 3x^2 - 1 \equiv 0 \pmod{8} \) and \( 3x^2 \equiv 1 \pmod{8} \)
Now by cases: if \( x = 2k+1 \),
then \( 3x^2 = 3(2k+1)^2 = 3[4k^2 + 4k + 1] = 3[4k(k+1) + 1] \equiv 3 \pmod{8} \), contrary to the equation.
And if \( x = 2k \),
then \( 3x^2 = 3(2k)^2 = 12k^2 \equiv 4k^2 \pmod{8} \) if \( k \) itself is even,
or \( 4k^2 \pmod{8} \equiv 4 \pmod{8} \) if \( k \) itself is odd, contrary to the equation.

9. Prove that there are no integers \( x \) and \( y \) such that \( x^4 - 16y^4 = 3 \).

Rewrite the equation as \( x^4 - 3 = 16y^4 \) and reduce both sides (mod 16):
Then \( x^4 - 3 \equiv 0 \pmod{16} \) and \( x^4 \equiv 3 \pmod{16} \)
Now by cases: if \( x = 2k+1 \), then exercise 3 shows \( x^4 \equiv 1 \pmod{16} \), contrary to the equation.
And if \( x = 2k \), then \( x^4 = (2k)^4 = 16k^4 \equiv 0 \pmod{16} \), contrary to the equation.

11. Based on some examples, we conjecture that if \( a \neq b \), then \( \frac{2ab}{a+b} < \sqrt{ab} \), for positive \( a \) and \( b \).
This inequality is equivalent to \( \frac{\sqrt{ab}}{a+b} < \frac{1}{2} \) by dividing both sides by \( 2\sqrt{ab} \).
This inequality is equivalent to \( \sqrt{ab} < \frac{a+b}{2} \) by multiplying both sides by \( (a+b) \).
This inequality was proved in Example 1 on page 215, so reversing the steps completes the proof.