25. \( P(n) \): A set with \( n \) elements has exactly \( n(n–1)/2 \) subsets with exactly two elements, for \( n \geq 2 \).

**Basis**: \( [n = 2] \) If the set has two elements \( \{a_1, a_2\} \), then it has only 1 two-element subset and \( 1 = 2(2–1)/2 \)

**Implication**: If a set with \( k \) elements has exactly \( k(k–1)/2 \) subsets with exactly two elements, then consider a set with \( (k+1) \) elements \( \{a_1, a_2, \ldots, a_k, a_{k+1}\} \).

Now there \( k \) sets of the form \( \{a_i, a_{k+1}\} \) for \( 1 \leq i \leq k \) that use the element \( a_{k+1} \) and by the induction hypothesis exactly \( k(k–1)/2 \) subsets with exactly two elements that do not use the element \( a_{k+1} \).

The total is \( k + \frac{k(k–1)}{2} = \frac{2k + k(k–1)}{2} = \frac{k(2k+1)(k+1)–1}{2} \), which shows \( P(k) \) implies \( P(k+1) \).

31. Let \( F \) be the number of five-cent and \( S \) be the number of six-cent stamps. Consider pairs \((F, S)\)

a) One stamp: \( (1, 0) = 5¢ \) and \( (0, 1) = 6¢ \)

b) \( P(n) \): A postage of \( n¢ \) can be formed from \( F \) and \( S \), for \( n \geq 20 \).

**Basis**: The value of \( (4, 0) \) is 20¢, so the basis is true.

**Implication**: If an amount of \( k¢ \) for \( k \geq 20 \) can be formed from \((F+S)\) stamps, then we show that an amount of \( (k+1)¢ \) can be formed by replacing one of the five-cent stamps by a six-cent stamp, if \( F \geq 1 \); or by replacing four six-cent stamps by five five-cent stamps, since otherwise there must be more than three six-cent stamps. Hence \( P(k) \) implies \( P(k+1) \).

C) \( P(n) \): A postage of \( n¢ \) can be formed from \( F \) and \( S \), for \( n \geq 20 \).

**Basis**: \( P(20), P(21), P(22), P(23), P(24) \) are true as shown above, using four stamps.

**Implication**: If \( P(j) \) is true for all \( j \), where \( 4 \leq j \leq k \) and \( k \geq 24 \), then \( (k+1) \geq 25 \) and \( (k–4) \geq 20 \), so by the strong induction hypothesis \( P(k–4) \) can be formed, and adding a single five-cent stamp gives \((k+1)¢\). Hence the strong hypothesis \([P(20) \lor P(21) \lor \ldots \lor P(k)]\) implies \( P(k+1) \).

33. An ATM machine dispenses \$20 and \$50 bills. Let \( T \) be the number of \$20 bills and \( F \) the number of \$50 bills. Then consider pairs \((T, F)\):

One bill: \( (1, 0) = \$20 \), \( (0, 1) = \$50 \)

Two bills: \( (2, 0) = \$40 \), \( (1, 1) = \$70 \), \( (0, 2) = \$100 \)

Three bills: \( (3, 0) = \$60 \), \( (2, 1) = \$90 \), \( (1, 2) = \$120 \), \( (0, 3) = \$150 \)

Four bills: \( (4, 0) = \$80 \), \( (3, 1) = \$110 \), etc.

\( P(n) \): Any amount of value \$10n can be formed for \( n \geq 4 \).

**Basis**: \( P(4), P(5), P(6) \) are true as shown above, using two, one, or three bills.

**Implication**: If \( P(j) \) is true for all \( j \), where \( 4 \leq j \leq k \) and \( k \geq 6 \), then \( (k+1) \geq 7 \) and \( (k–2) \geq 4 \), so by the strong induction hypothesis \( P(k–2) \) is true which means \$10k – \$20 can be formed; and replacing a \$20 by a single \$50 will increase that amount to \$10k – \$20 + (\$50 – \$20) = \$10k + \$10 = \$10(k+1).

Hence the strong hypothesis \([P(4) \lor P(5) \lor \ldots \lor P(k)]\) implies \( P(k+1) \).

41. \( P(n) \): \( D_x[x^n] = n\cdot x^{n–1} \) for \( n \geq 1 \).

**Basis**: \( D_x[x^1] = 1 = 1\cdot x^{(1–1)} \)

**Implication**: If \( P(k) \) is true, \( D_x[x^k] = k\cdot x^{(k–1)} \), so \( D_x[x^{(k+1)}] = D_x[x\cdot x^n] = [1\cdot x^n + (x)\cdot n\cdot x^{(n–1)}] = (n+1)\cdot x^n \)

So \( P(k) \) implies \( P(k+1) \).

43, 51, 53. [Solutions not shown.]